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Auxiliary matrices for the six-vertex model and the algebraic Bethe ansatz

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Abstract

We connect two alternative concepts of solving integrable models, Baxter's method of auxiliary matrices (or Q-operators) and the algebraic Bethe ansatz. The main steps of the calculation are performed in a general setting and a formula for the Bethe eigenvalues of the Q-operator is derived. A proof is given for states which contain up to three Bethe roots. Further evidence is provided by relating the findings to the six-vertex fusion hierarchy. For the XXZ spin-chain we analyse the cases when the deformation parameter of the underlying quantum group is evaluated both at and away from a root of unity.

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1. Introduction

This paper is a continuation of two previous works [1, 2] on the six-vertex model and the associated XXZ Heisenberg spin-chain at roots of unity. That is, we consider the integrable model defined via the Hamiltonian

$$H = \sum_{m=1}^{M} \sigma_m^+ \sigma_{m+1}^- + \sigma_m^- \sigma_{m+1}^+ + \frac{q+q^{-1}}{4} \sigma_m^z \sigma_{m+1}^z \qquad \sigma_{M+1}^{\pm} \equiv \lambda^{\pm 2} \sigma_1^{\pm} \quad \sigma_{M+1}^z \equiv \sigma_1^z. \tag{1}$$

Here $\{\sigma^x, \sigma^y, \sigma^z\}$ are the Pauli matrices with $\sigma^{\pm} = (\sigma^x \pm \mathrm{i}\sigma^y)/2$ and $\lambda \in \mathbb{C}$ fixes the boundary conditions. The anisotropy parameter in front of the third term in (1) is fixed in terms of the complex variable q. Of particular interest to our discussion is the case when q is a primitive root of unity, i.e. $q^N = 1$ for some integer $N > 2$ (we exclude here the cases $N = 1, 2$ which are related to the XXX model). At these particular values the above Hamiltonian and the associated six-vertex transfer matrix exhibit extra degeneracies in their spectra which are linked to an underlying infinite-dimensional non-Abelian symmetry algebra. At periodic boundary conditions, $\lambda = 1$, and in the commensurate sectors $2S^z = 0 \bmod N$ this symmetry algebra is isomorphic to the loop algebra of sl_2. This has been first established by Deguchi *et al* in [3], where additional results for $N = 3, 4$ outside the commensurate sectors can be found. When $\lambda \neq 1$ the symmetry algebra will in general reduce to the upper or lower Borel subalgebra [4, 5]. A special case is obtained when the boundary conditions are tuned to $\lambda = q^{\pm S^z}$. Then the symmetry algebra and the explicit form of its generators are known for all spin sectors and integers N [5].

The presence of a non-Abelian symmetry algebra is of interest as it now allows one to connect the Bethe ansatz and representation theory. Note that while this idea is not new, the mentioned symmetries are infinite dimensional and exist at finite length of the chain, $M < \infty$. The combination of these two distinct features distinguishes the present discussion from previously considered cases in the literature at open boundary conditions [6] or at infinite volume [7].

An important first step in connecting the Bethe ansatz with the representation theory of the aforementioned symmetry algebras is to find an efficient way to analyse the structure of the degenerate eigenspaces of the Hamiltonian as well as the transfer matrix. This can be achieved by using Baxter's concept of auxiliary matrices [8], also known as Q-operators, which satisfy certain operator functional equations with the transfer matrix. The concept of auxiliary matrices has primarily received attention in the context with the eight-vertex model [9–12]; see [13, 14] for a recent discussion corresponding to the root of unity case.

In connection with the six-vertex model the subject has obtained fresh impetus from new methods of constructing such Q-operators which provide an alternative to Baxter's procedure described in, e.g., [15]. These new methods are based on the representation theory of quantum groups [16–19, 1]. The latter method will be of importance to us as the auxiliary matrices constructed in this manner, see [1, 2], have been shown for several examples to yield information on the irreducible representations of the symmetry algebra at roots of unity. Several facts about the spectrum of the auxiliary matrices in [2] have so far only been rigorously proved for $N = 3$ and conjectured to hold true for $N > 3$ employing numerical calculations.

The purpose of this paper is to lend further support to the earlier conjectures and to extend the discussion from auxiliary matrices with periodic boundary conditions ($\lambda = 1$) to quasi-periodic ones ($\lambda \neq 1$) in order to accommodate the findings in [5]. To this end we take a broader point of view and consider the connection between the algebraic Bethe ansatz [20] and auxiliary matrices in a general setting and at generic values of q.

Away from a root of unity this will enable us to calculate the spectrum of the auxiliary matrices constructed in [19] and resolve certain convergence problems due to an infinite-dimensional auxiliary space. We will also make contact with the discussion in [17].

At roots of unity [1, 2] the Bethe ansatz analysis will not yield the complete set of eigenvalues for the Q-operators, as at most the highest weight state in each degenerate eigenspace of the transfer matrix ought to be a proper Bethe state, i.e. a state parametrized by finite solutions of the Bethe ansatz equations. Detailed explanations will be given in the text. In addition to the comparison with the algebraic Bethe ansatz we will also make contact with the fusion hierarchy of the six-vertex model. The latter provides an infinite series of higher-spin transfer matrices which can successively be generated through a functional equation. At roots of unity this series truncates and we will show how the auxiliary matrices are related to the fusion matrices, similar to the discussion in [14] for the eight-vertex model. This will provide additional evidence for the spectrum of the auxiliary matrices constructed in [1, 2].

The paper is organized as follows. In section 2 we introduce the monodromy matrices of the six-vertex transfer matrix and the Q-operators constructed in [19, 1, 2]. Afterwards we derive the commutation relations between the auxiliary matrices and the Yang–Baxter algebra.

This sets the stage to compute the action of the auxiliary matrices on Bethe states in section 3. We will find that they are eigenstates of the auxiliary matrices and compute the corresponding eigenvalues. This is done without specifying the quantum space. The computations are quite lengthy and cumbersome whence we will only give a proof for Bethe states which contain up to three Bethe roots. For the general case of an arbitrary number of Bethe roots we formulate a conjecture.

In section 4 we specialize our findings to the XXZ spin-chain. While we discuss both q being a root of unity and q not being a root of unity, the former case is considered in more detail in light of the aforementioned symmetries. We check the conjectured formula for the eigenvalues of the Q-operators for consistency by inserting them into the respective functional equation with the transfer matrix.

In section 5 we present further support for the conjecture regarding the spectrum of the auxiliary matrices by showing that the eigenvalues are also consistent with the fusion hierarchy of the six-vertex model. The fusion hierarchy is solved explicitly in terms of Bethe roots and we derive its 'truncation' at roots of unity.

Section 6 contains the conclusions.

2. Monodromy matrices

The first step in the application of the algebraic Bethe ansatz is the definition of the monodromy matrices. The monodromy matrices act on a tensor product of two spaces, $\mathcal{H}_0 \otimes \mathcal{H}$; the first is called the auxiliary space and the second the quantum space. Depending on the choice of the auxiliary space we obtain the monodromy matrices for the Q-operator and the six-vertex transfer matrix. Since the monodromy matrices are subject to the Yang–Baxter equation they can be constructed in the framework of quantum groups. Let $\{e_i, f_i, q^{h_i}\}_{i=0,1}$ be the Chevalley–Serre generators of the quantum loop algebra $U_q(\widetilde{sl}_2)$ subject to the relations

$$q^{h_i} e_j q^{-h_i} = q^{\mathcal{A}_{ij}} e_j \qquad q^{h_i} f_j q^{-h_i} = q^{-\mathcal{A}_{ij}} f_j \qquad q^{h_i} q^{h_j} = q^{h_j} q^{h_i} \qquad i, j = 0, 1 \tag{2}$$

where the Cartan matrix reads

$$\mathcal{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

We will only be dealing with representations where the central charge of the affine extension is zero, whence we set

$$h \equiv h_1 = -h_0. \tag{3}$$

In addition, for $i \neq j$ the Chevalley–Serre relations hold,

$$\begin{aligned} &e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0 \\ &f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0. \end{aligned} \tag{4}$$

The quantum algebra can be made into a Hopf algebra, its most important property being that of a coproduct which we choose to be ($i = 0, 1$)

$$\Delta(e_i) = 1 \otimes e_i + q^{h_i} \otimes e_i \qquad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i \qquad \Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i}. \tag{5}$$

The opposite coproduct Δ^{op} is obtained by permuting the two factors. If the deformation parameter q is considered to be an abstract indeterminate there exists the universal R-matrix intertwining these two coproduct structures

$$\mathbf{R}\Delta(x) = \Delta^{\text{op}}(x)\mathbf{R} \qquad x \in U_q(\widetilde{sl}_2) \qquad \mathbf{R} \in U_q(b_+) \otimes U_q(b_-). \tag{6}$$

Here $U_q(b_\pm)$ denote the upper and lower Borel subalgebra, respectively. We now define the six-vertex monodromy matrix by setting

$$\mathbf{T}(z) = \left(\pi_z^{(1)} \otimes \pi_{\mathcal{H}}\right)\lambda^{h\otimes 1}\mathbf{R} \in \operatorname{End}\mathbb{C}^2 \otimes \mathcal{H} \tag{7}$$

with $\pi_z^{(1)} : U_q(\widetilde{sl}_2) \to \operatorname{End}\mathbb{C}^2$ being the fundamental evaluation representation given in terms of Pauli matrices,

$$\begin{array}{lll} \pi_z^{(1)}(e_0) = z\sigma^- & \pi_z^{(1)}(f_0) = z^{-1}\sigma^+ & \pi_z^{(1)}(q^{h_0}) = q^{-\sigma^z} \\ \pi_z^{(1)}(e_1) = \sigma^+ & \pi_z^{(1)}(f_1) = \sigma^- & \pi_z^{(1)}(q^{h_1}) = q^{\sigma^z}. \end{array} \tag{8}$$

We will denote the basis in the corresponding representation space $\mathbb{C}^2$ by $\{|0\rangle, |1\rangle\}$. The representation $\pi_{\mathcal{H}}$ in the second factor determines the quantum space of our theory. At the moment we leave it unspecified in order to emphasize the general nature of the following discussion. Later we will set $\mathcal{H} = (\mathbb{C}^2)^{\otimes M}$ and $\pi_{\mathcal{H}} = \bigotimes_{m=1}^{M} \pi_{\zeta_m}^{(1)}$ in order to obtain the inhomogeneous XXZ spin-chain. The factor in front of the universal R-matrix involving λ fixes the boundary conditions.

In the context of the algebraic Bethe ansatz it is customary to decompose the monodromy matrix w.r.t. the auxiliary space $\mathbb{C}^2$, i.e. one introduces the following elements of $\operatorname{End}\mathcal{H}$:

$$A = \langle 0|\mathbf{T}|0\rangle_{\mathbb{C}^2} \qquad B = \langle 0|\mathbf{T}|1\rangle_{\mathbb{C}^2} \qquad C = \langle 1|\mathbf{T}|0\rangle_{\mathbb{C}^2} \qquad D = \langle 1|\mathbf{T}|1\rangle_{\mathbb{C}^2}. \tag{9}$$

Here the subscript $\mathbb{C}^2$ indicates that the matrix elements are taken w.r.t. the first factor in (7). These elements obey the familiar commutation relations of the six-vertex Yang–Baxter algebra which are deduced from the relation

$$R_{12}(w/z)\mathbf{T}_1(w)\mathbf{T}_2(z) = \mathbf{T}_2(z)\mathbf{T}_1(w)R_{12}(w/z) \tag{10}$$

with

$$R(z, q) = \frac{a+b}{2} 1 \otimes 1 + \frac{a-b}{2}\sigma^z \otimes \sigma^z + c\sigma^+ \otimes \sigma^- + c'\sigma^- \otimes \sigma^+ \tag{11}$$

denoting the six-vertex R-matrix. The parametrization of the Boltzmann weights is chosen in accordance with (5) and (8),

$$a = 1 \qquad b = \frac{1-z}{1-zq^2}q \qquad c = \frac{1-q^2}{1-zq^2} \qquad c' = cz. \tag{12}$$

We next define the monodromy matrix from which we will obtain below the auxiliary matrix or Q-operator. We follow an analogous procedure to that just given with the difference that we now change to a higher-dimensional auxiliary space $\mathcal{H}_0$ belonging to a suitably chosen representation $\pi'_w : U_q(b_+) \to \operatorname{End}\mathcal{H}_0$. The auxiliary space $\mathcal{H}_0$ will turn out to be finite dimensional at $q^N = 1$ and infinite dimensional when q is not a root of unity. The monodromy matrix is now

$$\mathbf{Q}(w) = (\pi'_w \otimes \pi_{\mathcal{H}})\lambda^{h\otimes \mathbf{1}}\mathbf{R} \in \operatorname{End}\mathcal{H}_0 \otimes \mathcal{H}. \tag{13}$$

At roots of unity there arises a technical subtlety. The universal R-matrix need not exist as whether one can construct an intertwiner depends on the precise nature of the representations $\pi'_w, \pi_{\mathcal{H}}$. This is due to the enlarged centre of the quantum group at roots of unity, see e.g. [21]. However, we will assume that the representations are always chosen such that this is the case. Then the definition (13) has to be replaced by this intertwiner which is to be explicitly constructed. (In the case of the monodromy matrix (7) one can first evaluate it away from a root of unity and then safely take the root of unity limit.)

Similar to the six-vertex monodromy matrix we may also decompose (13) over the auxiliary space,

$$\mathbf{Q} = (Q_{ij}) \quad \text{with} \quad Q_{ij} := \langle i|\mathbf{Q}|j\rangle_{\mathcal{H}_0} \in \operatorname{End}\mathcal{H} \tag{14}$$

where the states are labelled by integers lying in a finite interval, $0 \leqslant i, j \leqslant N'-1$, at roots of unity $q^{N'} = \pm 1$ or in an infinite interval $i, j \in \mathbb{Z}$ when $q^{N'} \neq \pm 1$. Here and in the following we set $N' = N$ if the order of the root of unity is odd and $N' = N/2$ when it is even.

We will specify the respective auxiliary spaces momentarily. Before doing so we introduce a third operator $L \in \mathrm{End}\,\mathcal{H}_0 \otimes \mathbb{C}^2$ which intertwines the tensor product $\pi'_w \otimes \pi_z^{(1)}$ and satisfies together with the monodromy matrices (7) and (13) the Yang–Baxter equation, i.e.

$$L_{12}(w/z)\mathbf{Q}_1(w)\mathbf{T}_2(z) = \mathbf{T}_2(z)\mathbf{Q}_1(w)L_{12}(w/z). \tag{15}$$

Here we have assumed that we are only allowing for representations π'_w in (13) for which

$$[L(w), \pi'_w(h) \otimes 1 + 1 \otimes \sigma^z] = 0. \tag{16}$$

This restriction is necessary in order to accommodate the quasi-periodic boundary conditions when $\lambda \neq 1$. Again, we decompose the L-operator, this time over the second factor, the two-dimensional space associated with the evaluation representation $\pi_z^{(1)}$,

$$L = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\delta} \end{pmatrix} \qquad \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathrm{End}\,\mathcal{H}_0. \tag{17}$$

We shall now explicitly specify the evaluation representations π'_w at and away from a root of unity as well as the matrix elements of the L-operator within these representations.

2.1. The auxiliary space when $q^N = 1$

We adopt the conventions in [2] and define the following evaluation representation $\pi'_w \equiv \pi^\mu_w$ of the full affine quantum algebra $U_q(\widetilde{sl}_2)$ at $q^N = 1$ or equivalently $q^{N'} = \pm 1$. Let $0 \leqslant n \leqslant N'-1$, $\mu \in \mathbb{C}$ and set

$$\begin{aligned}
&\pi^\mu_w(f_1)|n\rangle = |n+1\rangle \qquad \pi^\mu_w(f_1)|N'-1\rangle = 0 \qquad \pi^\mu_w(e_0) = w\pi^\mu_w(f_1) \\
&\pi^\mu_w(e_1)|n\rangle = \frac{\mu + \mu^{-1} - \mu q^{2n} - \mu^{-1} q^{-2n}}{(q-q^{-1})^2}|n-1\rangle \qquad \pi^\mu_w(f_0) = w^{-1}\pi^\mu_w(e_1) \\
&\pi^\mu_w(q^{h_1})|n\rangle = \mu^{-1}q^{-2n-1}|n\rangle \qquad \pi^\mu_w(q^{h_0}) = \pi^\mu_w(q^{-h_1}).
\end{aligned} \tag{18}$$

The non-vanishing matrix elements of the L-operator then read [1]

$$\begin{aligned}
&\alpha_n = \langle n|\boldsymbol{\alpha}|n\rangle = (w/z)\mu^{-\frac{1}{2}}q^{-n+\frac{1}{2}} - \mu^{\frac{1}{2}}q^{n+\frac{1}{2}} \\
&\delta_n = \langle n|\boldsymbol{\delta}|n\rangle = (w/z)\mu^{\frac{1}{2}}q^{n+\frac{3}{2}} - \mu^{-\frac{1}{2}}q^{-n-\frac{1}{2}} \\
&\gamma_n = \langle n|\boldsymbol{\gamma}|n+1\rangle = \mu^{\frac{1}{2}}q^{n+\frac{3}{2}}\frac{\mu + \mu^{-1} - \mu q^{2n+2} - \mu^{-1}q^{-2n-2}}{q-q^{-1}} \qquad n < N'-1 \\
&\beta_n = \langle n|\boldsymbol{\beta}|n-1\rangle = (w/z)(q-q^{-1})\mu^{-\frac{1}{2}}q^{-n+\frac{1}{2}} \qquad n > 0.
\end{aligned} \tag{19}$$

Note that this representation can be extended to generic q but is then infinite dimensional, i.e. $n \in \mathbb{Z}_{\geqslant 0}$. The reason for this particular choice of the representation is explained in [1, 2]. Here we simply recall that the following exact sequence holds [1]:

$$0 \to \pi^{\mu q}_{w'} \overset{\iota}{\hookrightarrow} \pi^\mu_w \otimes \pi^{(1)}_z \overset{\tau}{\to} \pi^{\mu q^{-1}}_{w''} \to 0 \qquad w = w'q^{-1} = w''q = z/\mu \tag{20}$$

with the inclusion ι and the projection τ detailed in [1] (cf section 4.1 and 4.2). From this decomposition of the tensor product one now derives a functional equation of the following type:

$$T(z)Q_\mu(w) = \phi_1(z)Q_{\mu q}(w') + \phi_2(z)Q_{\mu q^{-1}}(w'') \tag{21}$$

where

$$T = \big(\operatorname*{Tr}_{\pi_z^{(1)}} \otimes 1_{\mathcal{H}}\big)\mathbf{T} = A + D \quad \text{and} \quad Q_\mu = \big(\operatorname*{Tr}_{\pi_w^\mu} \otimes 1_{\mathcal{H}}\big)\mathbf{Q} = \sum_{n=0}^{N'} Q_{nn}. \tag{22}$$

Here ϕ_1, ϕ_2 are some coefficient functions whose precise form depends on the choice of the quantum space $\mathcal{H}$ which is as yet unspecified. For the XXZ spin-chain we will present them below.

2.2. *Infinite-dimensional auxiliary space for $q^N \neq 1$*

Rossi and Weston introduced in [19] the following infinite-dimensional four-parameter representation $\pi^+ = \pi^+(w; s_0, s_1, s_2)$ of the upper Borel subalgebra $U_q(b_+)$:

$$\begin{gathered} \pi^+(e_1)|n\rangle = |n-1\rangle \qquad \pi^+(q^{h_1})|n\rangle = s_0 q^{-2n}|n\rangle \qquad \pi^+(q^{h_0}) = \pi^+(q^{-h_1}) \\ \pi^+(e_0)|n\rangle = \left(s_1 s_2 \frac{(q-q^{-1})^2}{w} + \frac{w}{(q-q^{-1})^2} + s_1 q^{2n} + s_2 q^{-2n}\right)|n+1\rangle. \end{gathered} \tag{23}$$

Here $n \in \mathbb{Z}$ and the representation space is thus infinite dimensional. For convenience we rescale the parameters according to

$$s_{1,2} \to r_{1,2} = -(q-q^{-1})^2 w^{-1} s_{1,2} \qquad s_0 \to r_0 = s_0. \tag{24}$$

The matrix elements for the intertwiner L are now calculated to be [19]

$$\begin{aligned} \alpha_n &= \langle n|\alpha|n\rangle = (w/z) r_2 r_0^{-\frac{1}{2}} q^{-n+2} - r_0^{-\frac{1}{2}} q^n \\ \delta_n &= \langle n|\delta|n\rangle = (w/z) r_1 r_0^{\frac{1}{2}} q^n - r_0^{\frac{1}{2}} q^{-n} \\ \gamma_n &= \langle n|\gamma|n+1\rangle = (q-q^{-1}) r_0^{-\frac{1}{2}} q^{n+1} \\ \beta_n &= \langle n|\beta|n-1\rangle = (w/z) r_0^{\frac{1}{2}} q^{-n+1} \frac{r_1 r_2 + 1 - r_1 q^{2n-2} - r_2 q^{-2n+2}}{q-q^{-1}}. \end{aligned} \tag{25}$$

Similar to the root of unity case one has a decomposition of the tensor product $\pi^+ \otimes \pi_z^{(1)}$ according to the exact sequence [19]

$$0 \to \pi^+(wq^2; \mathbf{r}^+) \overset{\iota}{\hookrightarrow} \pi^+(w; \mathbf{r}) \otimes \pi_z^{(1)} \overset{\tau}{\to} \pi^+(wq^{-2}; \mathbf{r}^-) \to 0 \qquad w = z \tag{26}$$

with

$$\mathbf{r} = (r_0, r_1, r_2) \quad \text{and} \quad \mathbf{r}^\pm = (r_0 q^{\pm 1}, r_1 q^{\mp 2}, r_2). \tag{27}$$

The inclusion and projection maps now are [19]

$$\iota|n\rangle^+ = r_0 \frac{q^{-2n+1} - r_1 q^{-1}}{q-q^{-1}} |n\rangle \otimes |0\rangle + |n-1\rangle \otimes |1\rangle \quad \text{and} \quad \tau|n+1\rangle \otimes |0\rangle = |n\rangle^-.$$

Again this representation theoretic fact is the platform for deriving a functional equation of the type

$$T(z)Q(z; \mathbf{r}) = \psi_1(z) Q(z; \mathbf{r}^+) + \psi_2(z) Q(z; \mathbf{r}^-) \tag{28}$$

where one now has

$$Q(w; \mathbf{r}) = \big(\operatorname*{Tr}_{\pi^+(w;\mathbf{r})} \otimes 1_{\mathcal{H}}\big)\mathbf{Q} = \sum_{n=-\infty}^{\infty} Q_{nn}. \tag{29}$$

The reader will have noted that the sum in the definition of the auxiliary matrix is now infinite. In fact, this can cause convergence problems. From a mathematical point of view one should therefore treat q as an abstract indeterminate (rather than a complex number) and view the matrix elements of (29) as formal power series in q. We will return to this point later when we calculate the spectrum of (29). (See also the discussion in [19].)

2.3. Commutation relations with the Yang–Baxter algebra

Having specified explicitly the matrix elements of the intertwiner (17) at and away from a root of unity we can exploit the Yang–Baxter equation (15) in order to derive the commutation relations of the matrix elements (14) with the generators (9) of the Yang–Baxter algebra. One finds

$$\alpha_k Q_{kl} A = \alpha_l A Q_{kl} + \gamma_{l-1} B Q_{kl-1} - \beta_k Q_{k-1l} C \tag{30}$$

$$\alpha_k Q_{kl} B = \delta_l B Q_{kl} + \beta_{l+1} A Q_{kl+1} - \beta_k Q_{k-1l} D \tag{31}$$

$$\delta_k Q_{kl} C = \alpha_l C Q_{kl} + \gamma_{l-1} D Q_{kl-1} - \gamma_k Q_{k+1l} A \tag{32}$$

$$\delta_k Q_{kl} D = \delta_l D Q_{kl} + \beta_{l+1} C Q_{kl+1} - \gamma_k Q_{k+1l} B. \tag{33}$$

Here we have suppressed the dependence on the spectral variables in the notation which for say the second identity is explicitly given by

$$Q_{kl}(w)B(z) = \frac{\delta_l(w/z)}{\alpha_k(w/z)} B(z) Q_{kl}(w) + \frac{\beta_{l+1}(w/z)}{\alpha_k(w/z)} A(z) Q_{kl+1}(w) - \frac{\beta_k(w/z)}{\alpha_k(w/z)} Q_{k-1l}(w) D(z).$$

Depending on the type of representation determining the auxiliary space of Q there are different boundary conditions. For example, if $q^{N'} = \pm 1$ then

$$\beta_0 = 0 \quad \Rightarrow \quad \alpha_0 Q_{0l} B = \delta_l B Q_{0l} + \beta_{l+1} A Q_{0l+1}$$
$$\beta_{N'} = 0 \quad \Rightarrow \quad \alpha_k Q_{kN'-1} B = \delta_{N'-1} B Q_{kN'-1} - \beta_k Q_{k-1N'-1} D.$$

Away from a root of unity the auxiliary space is infinite dimensional and there are no boundary conditions.

Note at this point the difference in the alternative construction procedures for auxiliary matrices. In order to obtain an auxiliary matrix which commutes with the transfer matrix we used the concept of intertwiners leading to (15). The latter implies the commutator $[T(z), Q(w)] = 0$ which corresponds to the following non-trivial equation in terms of the Yang–Baxter algebra:

$$\sum_k \left(\frac{\gamma_{k-1}}{\alpha_k} B Q_{kk-1} - \frac{\gamma_k}{\delta_k} Q_{k+1k} B \right) = \sum_k \left(\frac{\beta_k}{\alpha_k} Q_{k-1k} C - \frac{\beta_{k+1}}{\delta_k} C Q_{kk+1} \right). \tag{34}$$

In contrast Baxter's method described, e.g., in [15] relies on the 'pair propagation through a vertex' property to construct an auxiliary matrix which commutes with T. The relation (15), which is the key to relating the Yang–Baxter algebra (9) with the matrix elements (14), is missing in Baxter's approach.

3. Action on Bethe states

Since the transfer matrix and the Q-operators commute, they must allow for a common set of eigenstates. This motivates us to compute the action of the Q-operator on the Bethe eigenstates of the transfer matrix, i.e. we now make the crucial assumption that the quantum space $\mathcal{H}$ contains a vector $|0\rangle_{\mathcal{H}}$, called the pseudo-vacuum, which satisfies

$$A|0\rangle_{\mathcal{H}} = |0\rangle_{\mathcal{H}}\langle 0|A|0\rangle_{\mathcal{H}} \qquad C|0\rangle_{\mathcal{H}} = 0 \qquad D|0\rangle_{\mathcal{H}} = |0\rangle_{\mathcal{H}}\langle 0|D|0\rangle_{\mathcal{H}} \tag{35}$$

and

$$Q_{kk}|0\rangle_{\mathcal{H}} = |0\rangle_{\mathcal{H}}\langle 0|Q_{kk}|0\rangle_{\mathcal{H}} \qquad Q_{jk}|0\rangle_{\mathcal{H}} = 0 \quad j > k. \tag{36}$$

In fact, when the quantum space $\mathcal{H}$ carries a representation $\pi_{\mathcal{H}}$ of the quantum group $U_q(\widetilde{sl}_2)$ or $U_q(b_-)$ this pseudo-vacuum is identified with a (unique) highest weight vector. The above properties then follow from the intertwining property (6) of the monodromy matrices. For instance, one has for $\mathbf{Q}$ that

$$[\mathbf{Q}, \pi'_w(q^h) \otimes \pi_{\mathcal{H}}(q^h)] = 0 \quad \Rightarrow \quad \pi_{\mathcal{H}}(q^h) Q_{jk} \pi_{\mathcal{H}}(q^{-h}) = q^{2(j-k)} Q_{jk}$$

which then implies (36) by exploiting the fact that $|0\rangle_{\mathcal{H}}$ is highest weight.

We now define a 'proper' Bethe eigenstate as a vector of the form

$$\left|z_1, \ldots, z_{n_B}\right\rangle_{\mathcal{H}} = \prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}} \tag{37}$$

where the parameters $z_j = z_j(q, \lambda)$ are *finite* solutions to the 'generalized' Bethe ansatz equations

$$\langle 0|A(z_i)|0\rangle_{\mathcal{H}} q^{n_B} P_B(z_i q^{-2}) + \langle 0|D(z_i)|0\rangle_{\mathcal{H}} q^{-n_B} P_B(z_i q^2) = 0. \tag{38}$$

Here we have introduced for convenience the 'Bethe polynomial'

$$P_B(z) = \prod_{j=1}^{n_B} (1 - z/z_j) \quad \text{with} \quad z_j = z_j(q, \lambda). \tag{39}$$

This set of coupled nonlinear equations is sufficient to guarantee that the Bethe states (37) are indeed eigenstates of the transfer matrix $T = A + D$. The proof follows the same lines as the well-known computation for the XXZ spin-chain [20]:

$$T(z) = \langle 0|A(z)|0\rangle_{\mathcal{H}} q^{n_B} \frac{P_B(zq^{-2})}{P_B(z)} + \langle 0|D(z)|0\rangle_{\mathcal{H}} q^{-n_B} \frac{P_B(zq^2)}{P_B(z)}. \tag{40}$$

We restrict ourselves to *finite* solutions of the Bethe ansatz equations (38) in order to take into account certain peculiarities which can occur at roots of unity. Some of the finite Bethe roots z_j at $q^{N'} \neq \pm 1$ can tend to zero or infinity when the root of unity limit is taken. In this limit it might also occur that a subset of Bethe roots $\{z_{i_\ell}\}_{\ell \in \mathbb{Z}_{N'}}$ forms a complete string,

$$\lim_{q' \to q} \prod_{\ell \in \mathbb{Z}_{N'}} \left(z - z_{i_\ell}(q')\right) = \prod_{\ell \in \mathbb{Z}_{N'}} \left(z - z_{i_0}(q) q^{2\ell}\right) = z^{N'} - z_{i_0}(q)^{N'} \qquad q^{N'} = \pm 1 \tag{41}$$

and so drops out of equation (38) [22]. In terms of the Yang–Baxter algebra the occurrence of a complete string corresponds to the vanishing of the following operator product:

$$\lim_{q' \to q} \prod_{\ell \in \mathbb{Z}_{N'}} B\left(z_{i_\ell}(q')\right) = 0 \qquad q^{N'} = \pm 1. \tag{42}$$

Hence, the Bethe states (37) evaluated at a root of unity might not yield a complete set of eigenstates. Moreover, the fact that the transfer matrix and the Q-operator possess a common set of eigenvectors does not necessarily imply that the Bethe states are also eigenstates of the Q-operator when degeneracies are present. This is precisely the case when $q^{N'} = \pm 1$. For instance, in [1], auxiliary matrices have been constructed for the six-vertex model at roots of unity which do not preserve the total spin and whose eigenstates therefore are different from the Bethe states. We will return to this discussion when we specialize our general set-up to the XXZ spin-chain. For the moment we keep the calculation as general as possible.

Let us start with the simplest case, only one 'Bethe root' is present. That is, we evaluate the B operator at solutions z_0 to the equation

$$\langle 0|A(z_0)|0\rangle_{\mathcal{H}} = \langle 0|D(z_0)|0\rangle_{\mathcal{H}}. \tag{43}$$

In order to compute the action of the Q-operator on the corresponding eigenstate of the transfer matrix we use the relation

$$Q_{kl}B = \left(\frac{\delta_l}{\alpha_k} - \frac{\beta_{l+1}\gamma_l}{\alpha_k\alpha_{l+1}}\right) B\, Q_{kl} + \frac{\beta_{l+1}}{\alpha_{l+1}} Q_{kl+1}A - \frac{\beta_k}{\alpha_k} Q_{k-1l}D + \frac{\beta_{l+1}\beta_k}{\alpha_{l+1}\alpha_k} Q_{k-1l+1}C \tag{44}$$

which follows from the identities (30), (31). Note that if $k = l$ all the rightmost operators on the right-hand side of the above equation possess the pseudo-vacuum as an eigenvector. Taking the trace in (44) we obtain (the boundary terms work out correctly if $q^{N'} = \pm 1$)

$$\sum_k Q_{kk}B = B \sum_k \left(\frac{\delta_k}{\alpha_k} - \frac{\beta_{k+1}\gamma_k}{\alpha_k\alpha_{k+1}}\right) Q_{kk} + \sum_k \frac{\beta_k}{\alpha_k} Q_{k-1k}(A - D) + \sum_k \frac{\beta_{k+1}\beta_k}{\alpha_k\alpha_{k+1}} Q_{k-1k+1}C.$$

When acting on the pseudo-vacuum the second and third terms on the RHS of the equation vanish, which leaves us with the eigenvalue

$$Q(w)B(z_0)|0\rangle_{\mathcal{H}} = \left\{\sum_k \left(\frac{\delta_k}{\alpha_k} - \frac{\beta_{k+1}\gamma_k}{\alpha_k\alpha_{k+1}}\right)\langle 0|Q_{kk}|0\rangle_{\mathcal{H}}\right\} B(z_0)|0\rangle_{\mathcal{H}}. \tag{45}$$

Here the sum runs over $\mathbb{Z}_{N'}$ when $q^{N'} = \pm 1$ and over $\mathbb{Z}$ when q is not a root of unity. The calculation for Bethe states with multiple Bethe roots follows the same logic, though it is now much more involved to show that all the 'unwanted' terms vanish due to the Bethe ansatz equations; see the appendix.

3.1. Conjecture

For a Bethe state with n_B Bethe roots we conjecture the following formula for the eigenvalues of the respective auxiliary matrices:

$$Q(w)\prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}} = \left\{\sum_k \langle 0|Q_{kk}(w)|0\rangle_{\mathcal{H}} \times \prod_{j=1}^{n_B}\left(\frac{\delta_k(w/z_j)}{\alpha_k(w/z_j)} - \frac{\beta_{k+1}(w/z_j)\gamma_k(w/z_j)}{\alpha_k(w/z_j)\alpha_{k+1}(w/z_j)}\right)\right\}\prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}}. \tag{46}$$

Note that the particular choice of the quantum space only enters via the pseudo-vacuum expectation value $\langle 0|Q_{kk}|0\rangle_{\mathcal{H}}$ and the Bethe roots $\{z_j\}_{j=1}^{n_B}$. The combination of matrix elements appearing in the product of the eigenvalue expression only depends on the auxiliary space. The above conjecture can be verified for states with $n_B = 2, 3$ by employing the intermediate steps detailed in the appendix. For $n_B > 3$ we will provide further support by showing that the eigenvalues satisfy the functional equations (21) and (28), respectively. For this purpose we need to determine the coefficient functions in (21) and (28) first. The latter depend on the choice of the quantum space and we now specialize in the XXZ spin-chain.

4. The XXZ spin-chain

As remarked upon earlier, one chooses for the familiar case of the inhomogeneous six-vertex model or XXZ spin-chain with quasi-periodic boundary conditions,

$$\pi_{\mathcal{H}} = \bigotimes_{m=1}^{M} \pi^{(1)}_{\zeta_m} \quad \text{and} \quad \mathcal{H} = (\mathbb{C}^2)^{\otimes M}. \tag{47}$$

Here $\{\zeta_m\}$ are some complex inhomogeneity parameters. The transfer and auxiliary matrix can then be written as the trace of an operator product

$$T(z) = \operatorname*{Tr}_{\pi_z^{(1)}} \lambda^{\sigma^z \otimes 1} R_{0M}(z/\zeta_M) \cdots R_{01}(z/\zeta_1) \tag{48}$$

and

$$Q(w) = \operatorname*{Tr}_{\pi'_w} \lambda^{\pi'_w(h) \otimes 1} L_{0M}(w/\zeta_M) \cdots L_{01}(w/\zeta_1). \tag{49}$$

Here R and L are the R-matrix and the L-operator specified earlier. Note again the difference in Baxter's construction procedure [15]. His final Q, which commutes with the transfer matrix, is in general not of this simple form. For the pseudo-vacuum and the associated 'expectation values' one finds

$$|0\rangle_{\mathcal{H}} = |0\rangle_{\mathbb{C}^2} \otimes \cdots \otimes |0\rangle_{\mathbb{C}^2} \qquad \langle 0|A|0\rangle_{\mathcal{H}} = \lambda \qquad \langle 0|D|0\rangle_{\mathcal{H}} = \lambda^{-1} q^M \prod_{m=1}^{M} \frac{z - \zeta_m}{zq^2 - \zeta_m} \tag{50}$$

and

$$\langle 0|\mathbf{Q}(w)|0\rangle_{\mathcal{H}} = \lambda^{\pi'_w(h') \otimes 1} \prod_{m=1}^{M} \alpha(w/\zeta_m) \quad \Rightarrow \quad Q_{nn}|0\rangle_{\mathcal{H}} = \lambda^{-2n} \prod_{m=1}^{M} \alpha_n(w/\zeta_m)|0\rangle_{\mathcal{H}}. \tag{51}$$

Here we have slightly modified our earlier conventions. Instead of taking the Cartan generator h in the exponent of the twist parameter, we introduced for convenience h' which in the two representations (18) and (23) is given by

$$\pi_w^{\mu}(h')|n\rangle = -2n|n\rangle \quad \text{and} \quad \pi^+(h')|n\rangle = -2n|n\rangle. \tag{52}$$

This simply amounts to a renormalization of the respective auxiliary matrices by an overall factor. We continue to treat the two cases of q being a root of unity and q not being a root of unity separately.

4.1. Roots of unity

When $q^{N'} = \pm 1$, the auxiliary space is finite and upon inserting expressions (19) into the conjectures formula (46) one obtains

$$\langle 0|Q_{nn}(w)|0\rangle_{\mathcal{H}} = \lambda^{-2n} \mu^{\frac{M}{2}} q^{Mn + \frac{M}{2}} \prod_{m=1}^{M} (w\mu^{-1} q^{-2n}/\zeta_m - 1) \tag{53}$$

and

$$Q_\mu(w) \prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}} = \Bigg\{ q^{S^z} \mu^{S^z} P_B(w\mu) P_B(w\mu^{-1}) \sum_{k \in \mathbb{Z}_{N'}} \times \frac{\lambda^{-2k} q^{2kS^z} \prod_{m=1}^{M} (w\mu^{-1} q^{-2k}/\zeta_m - 1)}{P_B(w\mu^{-1} q^{-2k}) P_B(w\mu^{-1} q^{-2k-2})} \Bigg\} \prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}}. \tag{54}$$

Here we have used the relation between total spin and the number of Bethe roots, $2S^z = M - 2n_B$. We emphasize that the Bethe roots z_j are assumed to be *finite* solutions to the Bethe ansatz equations *at* a root of unity. This is different from the polynomial which is obtained by solving the Bethe ansatz equations away from a root of unity and then taking the root of unity limit. As pointed out before, in this limit one might encounter vanishing or infinite Bethe roots as well as complete strings. Before we investigate these issues we first check the eigenvalues of the auxiliary matrix for consistency by making contact with the functional equation (21).

4.1.1. The TQ-equation. Along the same lines as detailed in section 5 of [1] for the case of periodic boundary conditions (see equations (111) and (112) therein), one now calculates for the twisted inhomogeneous XXZ spin-chain the following coefficient functions in (21):

$$T(z)Q_\mu(z/\mu) = \lambda^{-1}q^{\frac{M}{2}}\left(\prod_{m=1}^{M}\frac{z-\zeta_m}{zq^2-\zeta_m}\right)Q_{\mu q}(zq/\mu) + \lambda q^{\frac{M}{2}}Q_{\mu q^{-1}}(zq^{-1}/\mu). \tag{55}$$

Here use has been made of the explicit form of the inclusion and projection map detailed in equations (88), (89) and (99), (100) of [1], respectively. Employing the known expression (40) for the eigenvalues of the transfer matrix within the framework of the algebraic Bethe ansatz [20],

$$T(z) = \lambda q^{n_B}\frac{P_B(zq^{-2})}{P_B(z)} + \lambda^{-1}\left(\prod_{m=1}^{M}\frac{z-\zeta_m}{zq^2-\zeta_m}\right)q^{M-n_B}\frac{P_B(zq^2)}{P_B(z)} \tag{56}$$

one verifies that the conjectured eigenvalues of the auxiliary matrix satisfy (55). Note that we have implicitly made the assumption that all Q-matrices in (55) commute with each other. For $\lambda = 1$ this has been proved in [1]. Using the explicit form of the intertwiners employed in this proof one verifies that the same holds true for quasi-periodic boundary conditions as long as $\lambda^N = 1$. This is sufficient to cover the symmetries investigated in [5].

4.1.2. The degeneracies at roots of unity $q^{N'} = \pm 1$. Recall from [3, 5] that due to the loop symmetry at a root of unity $q^{N'} = \pm 1$ the eigenspaces of the transfer matrix are organized into multiplets containing states whose total spin S^z varies by multiples of N'. The Bethe states (37) with finite Bethe roots are assumed to correspond to the highest weight states in such multiplets, i.e. acting with the generators of the respective symmetry algebra on this state one successively obtains the whole degenerate eigenspace of the transfer matrix (see [23, 24, 2] for examples of the highest weight property). The states within a degenerate multiplet are *not* of the form (37), whence our result (54) only applies to the highest weight vectors containing finite Bethe roots. Therefore, it does not yield the complete spectrum of the auxiliary matrices Q_μ. We now compare for these highest weight states the identity (54) with the formulae (11) and (19) in [2] which have been proved for $N = 3$ using functional relations and conjectured to hold true for $N > 3$ based on numerical data.

Recall from [2] that the eigenvalues of the auxiliary matrices Q_μ for the homogeneous chain, $\{\zeta_m = 1\}_{m=1}^{M}$, and periodic boundary conditions, $\lambda = 1$, were shown to be of the following general form (see formulae (11) and (19) in [2]):

$$Q_\mu(w = z/\mu) = \mathcal{N}_\mu z^{\bar{n}_\infty}P_B(z)P_B(z\mu^{-2})P_S(z^{N'}, \mu^{2N'}). \tag{57}$$

In order to match our result (54) from the algebraic Bethe ansatz computation with the formulae (11) and (19) in [2] we have to identify

$$\mathcal{N}_{\mu=1}P_S(z^{N'}, \mu = 1) = \mathcal{N}_{\mu=1}\prod_{j=1}^{n_S}(1 - z^{N'}/a_j) = q^{S^z}\sum_{k\in\mathbb{Z}_{N'}}\frac{q^{2kS^z}(zq^{-2k}-1)^M}{P_B(zq^{-2k})P_B(zq^{-2k-2})}. \tag{58}$$

Note that in our derivation of (54) we have assumed infinite and vanishing Bethe roots to be absent. Thus, we have to set $\bar{n}_\infty = 0$ in formula (19) of [2]. Numerical evidence suggests that this is only true in the commensurate sectors $2S^z = 0 \bmod N$. One then finds for $N = 3$ agreement between the results in [2] and our present calculation up to a trivial redefinition of the normalization constant $\mathcal{N}_\mu$. Recall from the discussion in [2] that the rational function (58) is in fact a polynomial[1] in the variable $z^{N'}$ whose roots a_j contain all the essential information on the irreducible representation of the loop algebra spanning the degenerate eigenspaces of the transfer matrix. The reader is referred to [2] for details.

In contrast to the transfer matrix the auxiliary matrices Q_μ stay non-degenerate and in addition to the eigenvalues corresponding to the highest weight state one also needs to compute the remaining eigenvalues of the auxiliary matrix within the multiplet. The examples explicitly worked out for $N = 3$ and $M = 5, 6, 8$ in [2] suggest the following picture:

Set $\lambda = 1$, $\{\zeta_m = 1\}_{m=1}^M$ and assume we have a multiplet in a commensurate sector, i.e. the highest weight state has spin $2S^z = 0 \bmod N$. Then the form of the eigenvalue (54) remains the same except for a change in the μ-dependence of the polynomial (58). The examples in [2] showed that this polynomial changes within the multiplet according to the following rule:

$$S^z: \qquad P_S(z^{N'}, \mu^{2N'}) = \prod_{j=1}^{n_S}(1 - z^{N'}\mu^{-2N'}/a_j) \quad \text{(highest weight state)}$$

$$S^z - N': \qquad P_S(z^{N'}, \mu^{2N'}) = (1 - z^{N'}/a_1)\prod_{j=2}^{n_S}(1 - z^{N'}\mu^{-2N'}/a_j)$$

$$S^z - 2N': \qquad P_S(z^{N'}, \mu^{2N'}) = (1 - z^{N'}/a_1)(1 - z^{N'}/a_2)\prod_{j=3}^{n_S}(1 - z^{N'}\mu^{-2N'}/a_j)$$

$$\vdots$$

$$-S^z: \qquad P_S(z^{N'}) = \prod_{j=1}^{n_S}(1 - z^{N'}/a_j) \quad \text{(lowest weight state).}$$

That is, as one steps through the multiplet the polynomial (58) successively looses factors of $\mu^{-2N'}$ until one reaches the lowest weight state, where it does not contain any μ-factor. We can confirm this picture by computing the eigenvalue corresponding to the lowest weight state. Invoking the transformation of the auxiliary matrix under spin reversal $\mathfrak{R} = \prod_{m=1}^M \sigma_m^x$ one finds (cf formula (39) in [2]),

$$Q_\mu(w = z/\mu)\prod_{j=1}^{n_B} C(z_j)\mathfrak{R}|0\rangle_{\mathcal{H}} = \text{const} \times \mathfrak{R}Q_{\mu^{-1}}(w = z/\mu)\prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}}$$

$$= \left\{ q^{-S^z}\mu^{S^z} P_B(z)P_B(z\mu^{-2}) \sum_{k\in\mathbb{Z}_{N'}} \lambda^{-2k} q^{2kS^z} \frac{\prod_{m=1}^M (zq^{-2k}/\zeta_m - 1)}{P_B(zq^{-2k})P_B(zq^{-2k-2})} \right\} \prod_{j=1}^{n_B} C(z_j)\mathfrak{R}|0\rangle_{\mathcal{H}}.$$

In accordance with the picture outlined before, the sum yielding the polynomial P_S is now independent of the parameter μ.

4.1.3. Infinite and vanishing Bethe roots. Besides the missing states within a degenerate multiplet of the transfer matrix, we are also missing those states which contain infinite or vanishing Bethe roots in the root of unity limit. A concrete example for the homogeneous chain, i.e. $\{\zeta_m = 1\}_{m=1}^M$, is given by setting $M = 5$, $S^z = 1/2$ and considering the subspace

[1] We briefly recall the argument for N odd. First, note that the Bethe ansatz equations (38) ensure that the rational function (58) is pole free in z. Hence, it must be a polynomial in z. Secondly, the function (58) is obviously invariant under the replacement $z \to zq^2$. Thus, any zero $z' \neq 0$ occurs inside a string $\{z'q^{2\ell}\}_{\ell\in\mathbb{Z}_{N'}}$ and gives rise to a factor $(z^{N'} - z'^{N'})$ in the root decomposition of the polynomial.

of momentum $P = 0$. There one finds for $2\Delta = q' + q'^{-1}$ with q' not a root of unity the following Bethe roots:

$$b_1 = b_2^{-1} = \tfrac{1}{4}\left(1 + \Delta - \sqrt{5 + \Delta(\Delta - 2)} + \sqrt{\left(1 + \Delta - \sqrt{5 + \Delta(\Delta - 2)}\right)^2 - 16}\right).$$

Here we have temporarily introduced the parametrization $b_i = q'(1-z_i)/(1-z_i q'^2)$ in order to accommodate the occurrence of infinite roots. Taking the root of unity limit $q' \to q = \mathrm{e}^{2\pi \mathrm{i}/3}$ one now computes

$$\lim_{q'\to q} T(z) = 1 + \left(\frac{q(1-z)}{1-zq^2}\right)^5 \qquad \lim_{q'\to q} b_1 = \lim_{q'\to q} b_2^{-1} = \mathrm{e}^{2\pi\,\mathrm{i}/3} = q \quad z_1 = 0 \quad z_2 = \infty.$$

Therefore, the *finite* solutions to the Bethe ansatz equations do not give a complete set of eigenstates at roots of unity. Note that it might even happen that the highest weight state in a multiplet contains infinite or vanishing Bethe roots.

4.2. The case when q is not a root of unity

A priori one might expect that things are now simpler than the root of unity case as the only degeneracies which occur in the spectrum of the transfer matrix are due to spin-reversal symmetry. However, the auxiliary space is now infinite and one has to deal with potential convergence problems when taking the trace in (29). This becomes evident if we insert expressions (25) into the conjectured formula for the eigenvalue expression of the auxiliary matrix. One obtains for the pseudo-vacuum expectation value

$$\langle 0|Q_{nn}(w)|0\rangle_{\mathcal{H}} = \lambda^{-2n} \prod_{m=1}^{M} \alpha_n(w/\zeta_m) = \lambda^{-2n} r_0^{-\frac{M}{2}} q^{n\frac{M}{2}} \prod_{m=1}^{M} (wr_2 q^{-2n+2}/\zeta_m - 1) \tag{59}$$

and the action of the auxiliary matrix on a Bethe state produces

$$Q(z; r_0, r_1, r_2) \prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}} = \Bigg\{ r_0^{-S^z} P_B(z) P_B(zr_1r_2) \times \sum_{\ell\in\mathbb{Z}} \frac{\lambda^{-2\ell} q^{2\ell S^z} \prod_{m=1}^{M} (zr_2 q^{-2\ell+2}/\zeta_m - 1)}{P_B(zr_2 q^{-2\ell+2}) P_B(zr_2 q^{-2\ell})} \Bigg\} \prod_{j=1}^{n_B} B(z_j)|0\rangle_{\mathcal{H}}.$$

The above expression for the eigenvalue might not be convergent. In fact, it was discussed in [19] for $r_1 = r_2 = 0$ and $\lambda = 1$ that the matrix elements of the Q-operator contain the formal power series

$$\delta\big(q^{2S^z}\big) = \sum_{\ell\in\mathbb{Z}} q^{2\ell S^z} \qquad S^z = \frac{M}{2} - n_B \tag{60}$$

in the deformation parameter q. If one removes this factor in the sector $S^z = 0$ by an *ad-hoc* renormalization the remaining auxiliary matrix can be identified with Baxter's expression (101) in [10]. See the discussion in section 5 of [19] for details. Obviously, our algebraic Bethe ansatz analysis of the auxiliary matrices reproduces these findings. In the limit $r_1, r_2 \to 0$ we obtain apart from the factor (60) and the constant $r_0^{-S^z}$, the Bethe polynomial (39) as eigenvalue as we should. However, the required renormalization is certainly an unsatisfactory answer to the convergence problem, in particular as the factor (60) is needed outside the spin-sector $S^z = 0$ in order to satisfy the functional equation (28) which for the XXZ spin-chain is calculated to

$$Q(z;r_0,r_1,r_2)T(z) = \lambda^{-1}q^{\frac{M}{2}}Q(zq^{-2};r_0q^{-1},r_1q^2,r_2) + \lambda q^{\frac{M}{2}}\left(\prod_{m=1}^{M}\frac{z-\zeta_m}{zq^2-\zeta_m}\right)Q(zq^2;r_0q,r_1q^{-2},r_2). \tag{61}$$

In order to obtain a convergent expression let us modify the auxiliary space by restricting it to an invariant subspace at special values of the parameters $r_{1,2}$.

4.2.1. Truncation of the auxiliary space. In order to establish convergence it would be helpful to restrict the sum in the eigenvalue expression of the auxiliary matrix either to the positive or negative integers and then choose the twist parameter appropriately. In order to achieve this we have to find values for the free parameters (r_0, r_1, r_2) such that the representation (23) contains an invariant subspace. To this end we need for some $m_o \in \mathbb{Z}$ the truncation condition

$$e_0|m_o\rangle_{\pi^+} = w\frac{r_1r_2+1-r_1q^{2m_o}-r_2q^{-2m_o}}{(q-q^{-1})^2}|m_o\rangle_{\pi^+} = 0. \tag{62}$$

This is easily achieved by setting

$$\text{either}\quad r_1 = q^{-2m_o}\quad\text{or}\quad r_2 = q^{2m_o}. \tag{63}$$

For simplicity let us choose $m_o = 0$ and $r_2 = 1$. Denote by $\pi^+_{\leqslant} = \pi^+_{\leqslant}(w;r_0,r_1)$ the irreducible subrepresentation of (23) which is obtained by restriction to the negative integers (including zero) and set

$$Q_{\leqslant}(w;r_0,r_1) = (\operatorname*{Tr}_{\pi^+_{\leqslant}}\otimes 1_{\mathcal{H}})\mathbf{Q} = \sum_{n=-\infty}^{0}Q_{nn}. \tag{64}$$

Since the parameter r_2 does not shift in the decomposition (26) this truncation is consistent with the functional equation (28), albeit we have to supplement the projection map τ by defining

$$\tau|0\rangle_{\pi^+_{\leqslant}(w;r_0,r_1)}\otimes|1\rangle_{\pi_z^{(1)}} \equiv \frac{r_0(r_1-1)}{q^2-1}|0\rangle_{\pi^+_{\leqslant}(wq^{-2};r_0q^{-1},r_1q^2)}\quad r_1\neq 1\quad w=z. \tag{65}$$

For the semi-infinite representation space we now obtain the following eigenvalue associated with a Bethe state (37):

$$Q_{\leqslant}(z;r_0,r_1) = \lambda^{-2}q^{2S^z}r_0^{-S^z}P_B(zr_1)P_B(z)\sum_{\ell=1}^{\infty}\lambda^{2\ell}q^{-2\ell S^z}\frac{\prod_{m=1}^{M}(zq^{2\ell}/\zeta_m-1)}{P_B(zq^{2\ell})P_B(zq^{2\ell-2})}. \tag{66}$$

Here we have denoted the eigenvalue by the same symbol as the operator. Unlike in the previous expression the sum now only ranges over the positive integers. This puts us in the position to prove absolute convergence of the series in (66). We treat the cases $|q| = 1$ and $|q|^{\pm 1} > 1$ separately.

Suppose q is of modulus one, then we deduce from

$$|q|=1:\quad \sum_{\ell=1}^{\infty}\left|\lambda^{2\ell}q^{-2\ell S^z}\frac{\prod_{m=1}^{M}(zq^{2\ell}/\zeta_m-1)}{P_B(zq^{2\ell})P_B(zq^{2\ell-2})}\right| \leqslant \frac{\prod_{m=1}^{M}(|z/\zeta_m|+1)}{\prod_{j=1}^{n_B}(1-|z/z_j|)^2}\sum_{\ell=1}^{\infty}|\lambda|^{2\ell} \tag{67}$$

that absolute convergence is guaranteed as long as $|\lambda| < 1$.

Now let $|q|^{\pm 1} > 1$ and set $q = \exp v$, $z = \exp 2u$, $\zeta_m = \exp 2\xi_m$. Rewriting

$$\sum_{\ell=1}^{\infty} \left| \lambda^{2\ell} q^{-2\ell S^z} \frac{\prod_{m=1}^{M} (zq^{2\ell}/\zeta_m - 1)}{P_B(zq^{2\ell}) P_B(zq^{2\ell-2})} \right|$$

$$= \sum_{\ell=1}^{\infty} \frac{2|\lambda|^{2\ell} |q|^{n_B} |z|^{\frac{M}{2} - n_B} \prod_{m=1}^{M} |\sinh(u - \xi_m + \ell v)| |\zeta_m|^{-\frac{1}{2}}}{\prod_{j=1}^{n_B} |\sinh(u - u_j + \ell v) \sinh(u - u_j + (\ell - 1)v)| / |z_j|}$$

$$\leqslant \text{const} \sum_{\ell=1}^{\infty} |\lambda|^{2\ell} |q|^{\pm 2\ell |S^z|}$$

we find absolute convergence for $|\lambda| < |q|^{\mp M/2}$. Thus, in summary we are left with the condition

$$|\lambda| < |q|^{\mp M/2} \qquad |q|^{\pm 1} \geqslant 1 \tag{68}$$

which now includes the case when q is of modulus one.

Having assured convergence we can now verify that the eigenvalues (66) satisfy the functional equation (61). Doing so we again implicitly made the assumption that the Q-operators in the functional equation commute with each other. Since the auxiliary space is still semi-infinite, the proof of commutation via the construction of the corresponding intertwiners (similar to the root of unity case) is less feasible. Instead we are going to exploit the completeness of the Bethe ansatz when q is not a root of unity.

Provided one accepts that the Bethe states (37) and their counterparts under spin-reversal

$$\mathfrak{R}\big|z_1, \ldots, z_{n_B}\big\rangle_{\mathcal{H}} \propto \prod_{j=1}^{n_B} C(z_j) \mathfrak{R}|0\rangle_{\mathcal{H}} \qquad \mathfrak{R}|0\rangle_{\mathcal{H}} = |1\rangle_{\mathbb{C}^2} \otimes \cdots \otimes |1\rangle_{\mathbb{C}^2} \tag{69}$$

form a complete set of eigenstates which span the whole quantum space $\mathcal{H}$, one deduces that the three-parameter family $\{Q_{\leqslant}(z; r_0, r_1)\}$ of auxiliary matrices can simultaneously be diagonalized as the Bethe roots only depend on q and λ. Hence, we must have $[Q_{\leqslant}(z; r_0, r_1), Q_{\leqslant}(w; r_0', r_1')] = 0$.

Note that the commutation of the Q-operators for arbitrary spectral parameters together with the explicit form of the matrix elements (25) implies that the eigenvalues (66) are polynomials of degree $\leqslant M$ in the spectral variable z. Unlike the root of unity case, however, the sum by itself is not a polynomial. One now has to include a factor P_B in front of the sum in order to cancel the simple poles of the denominator at $z = z_j$. Employing the Bethe ansatz equations (38) one then deduces that for any closed contour C_ℓ around $z_j q^{-2\ell}$ with integer $\ell \geqslant 0$ one has

$$\oint_{C_\ell} \mathrm{d}z\, P_B(z) \sum_{\ell=1}^{\infty} \lambda^{2\ell} q^{-2\ell S^z} \frac{\prod_{m=1}^{M} (zq^{2\ell}/\zeta_m - 1)}{P_B(zq^{2\ell}) P_B(zq^{2\ell-2})} = 0.$$

The vanishing of the contour integral signals that the above rational function is indeed a polynomial in z. This is consistent with (66), where the parameter r_1 in the argument of the first factor is arbitrary.

For completeness we briefly discuss the transformation under spin-reversal in order to obtain the eigenvalues of the auxiliary matrix 'beyond the equator', i.e. for the states (69). Acting with the spin-reversal operator on the auxiliary matrix we obtain

$$\mathfrak{R} Q_{\leqslant}(z; r_0, r_1) \mathfrak{R} = \operatorname*{Tr}_{\pi^+_{\leqslant}} \lambda^{\pi^+_{\leqslant}(h') \otimes 1} \sigma_M^x L_{0M}(z/\zeta_M) \sigma_M^x \cdots \sigma_1^x L_{01}(z/\zeta_1) \sigma_1^x$$

which amounts in terms of the intertwiner L to the replacement

$$\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\} \to \{\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha}\}. \tag{70}$$

From (25) one derives the following identities for the matrix elements of the intertwiner L:

$$\begin{aligned}\alpha_n(wq^{-2}, q; r_0, r_1, r_2) &= \delta_n\big(w, q^{-1}; r_0^{-1}, r_2, r_1\big)\\ \delta_n(wq^{-2}, q; r_0, r_1, r_2) &= \alpha_n\big(w, q^{-1}; r_0^{-1}, r_2, r_1\big)\\ \beta_n\gamma_{n-1}|_{(wq^{-2},q;r_0,r_1,r_2)} &= \beta_n\gamma_{n-1}|_{(w,q^{-1};r_0^{-1},r_2,r_1)}.\end{aligned} \tag{71}$$

Note that only the above combinations of matrix elements contribute to the trace of the monodromy matrix (13), i.e. the auxiliary matrix (49). Hence, we have the identity

$$\mathfrak{R}Q_{\leqslant}(zq^{-2}, q; r_0, r_1, r_2 = 1)\mathfrak{R} = Q_{\leqslant}\big(z, q^{-1}; r_0^{-1}, r_2 = 1, r_1\big). \tag{72}$$

Thus, spin-reversal leads to the consideration of the truncated auxiliary matrix with $r_1 = 1$ and r_2 arbitrary. We now discuss this case in the context of q-oscillator representations. Before we have the following:

Remark. In order to ensure convergence for, for example, $|q| = 1$, we have excluded the quasi-periodic boundary conditions with $|\lambda| \geqslant 1$. However, in order to recover, for example, periodic boundary conditions, $\lambda = 1$, one may proceed as follows. Employing (61) one can express the transfer matrix eigenvalues in terms of the auxiliary matrix at $|\lambda| < 1$. While at the moment we do not have an analytic argument due to the implicit dependence of the Bethe roots on λ, numerical calculations suggest that one can then safely take the limit $\lambda \to 1$ in order to recover the eigenvalues of the six-vertex transfer matrix with periodic boundary conditions.

4.2.2. q-oscillator representations. We now simplify the representation (23) of [19] further in order to make contact with the results in [17] for the Coulomb gas formalism of conformal field theory. We explicitly show that one can derive the analogue of the functional equations therein also for the XXZ spin-chain.

The q-oscillator algebra is defined in terms of the generators [26]

$$qe_+e_- - q^{-1}e_-e_+ = (q - q^{-1})^{-1} \quad \text{and} \quad [h, e_\pm] = \pm 2e_\pm. \tag{73}$$

We define two representations $\varrho_\pm$ by identifying $e_0 \to e_\pm, e_1 \to e_\mp, h_1 = -h_0 \to \pm h$ and setting

$$\varrho_+ = \pi^+(w; r_0 = 1, r_1 = 1, r_2 = 0) \quad \text{and} \quad \varrho_- = \pi^+(w; r_0 = 1, r_1 = 0, r_2 = 1). \tag{74}$$

Again we note that the representation space in (23) truncates and it will be understood that the representation spaces associated with $\varrho_\pm$ are the modules obtained by acting freely with $e_\pm$ on the highest (lowest) weight vector $|0\rangle$. Let $Q^\pm = (\mathrm{Tr}_{\varrho\pm} \otimes 1)\mathbf{Q}$ then we find for the eigenvalues associated with a Bethe vector (37),

$$Q^+(z) = (-1)^M P_B(z) \sum_{\ell=0}^{\infty} \lambda^{2\ell} q^{-2\ell S^z} = \frac{(-1)^M P_B(z)}{1 - \lambda^2 q^{-2S^z}} \tag{75}$$

and

$$Q^-(z) = P_B(z) \sum_{\ell=0}^{\infty} \lambda^{2\ell} q^{-2\ell S^z} \frac{\prod_{m=1}^{M}(zq^{2\ell+2}/\zeta_m - 1)}{P_B(zq^{2\ell+2})P_B(zq^{2\ell})}. \tag{76}$$

For absolute convergence we have assumed that (68) holds as before. Note that in order to satisfy the functional equation (61) with the transfer matrix we have to keep in mind that the operators $Q^\pm$ still depend implicitly on the parameters (r_0, r_1, r_2) which shift in (26). This is particularly important in the case of $Q^+ = Q_{\leqslant}(r_0 = 1, r_1 = 1, r_2 = 0)$ where in the functional equation the shift $r_1 \to r_1 q^{\pm 2}$ occurs and, hence, the truncation condition (62) for the auxiliary space changes.

This highlights the importance of introducing the free parameters (r_0, r_1, r_2) in (23). Even if one would directly start the discussion with the simpler looking representations (74) the decomposition of the tensor product (26), which underlies the TQ-equation (28), leads to the consideration of the more general representation (23) (see also the comments in the introduction of [1] for the root of unity case). We should therefore understand expressions (75) and (76) as a decomposition of the general solution (66). Namely, we can write (66) as a product of (75) and (76) (compare with formula (4.10) in [17]),

$$Q_{\leqslant}(z; r_0, r_1) = (-1)^M r_0^{-S^z}(1 - \lambda^2 q^{-2S^z})Q^+(zr_1)Q^-(z). \tag{77}$$

One now easily derives the analogue of the functional relations reported in [17] for the Coulomb gas formalism. For instance, we find for the eigenvalues the following formula corresponding to the 'quantum Wronskian condition' (cf equation (4.3) in [17]):

$$Q^+(zq^2)Q^-(z) - \lambda^2 q^{-2S^z} Q^+(z)Q^-(zq^2) = \frac{\prod_{m=1}^M (1 - zq^2/\zeta_m)}{1 - \lambda^2 q^{-2S^z}}. \tag{78}$$

This is a special case of a more general relation (see equation (89) below) involving the fusion matrices of the six-vertex model, which we discuss next.

5. The six-vertex fusion hierarchy

In order to provide further support for our algebraic Bethe ansatz computation of the eigenvalues of the auxiliary matrices, we now make contact with what in the literature is known as the fusion hierarchy. We follow a similar line of argument to that in [14] for the eight-vertex model. First, we will briefly review the representation theoretic aspects in the construction of the fusion hierarchy which will be the key to connecting fusion and auxiliary matrices at roots of unity. We specialize at once to the XXZ spin-chain (47).

We start by introducing the monodromy matrices associated with the fusion matrices. Denote by $\pi_z^{(n)} : U_q(\widetilde{sl}_2) \to \mathrm{End}\,\mathbb{C}^{n+1}$ the spin $n/2$ evaluation representation of the quantum loop algebra, i.e.

$$\begin{aligned}
&\pi_z^{(n)}(e_1)|m\rangle = [n-m+1]_q|m-1\rangle && \pi_z^{(n)}(f_0) = z^{-1}\pi_z^{(n)}(e_1)\\
&\pi_z^{(n)}(f_1)|m\rangle = [m+1]_q|m+1\rangle && \pi_z^{(n)}(e_0) = z\pi_z^{(n)}(f_1)\\
&\pi_z^{(n)}(q^{h_1})|m\rangle = q^{n-2m}|m\rangle && \pi_z^{(n)}(q^{h_0}) = \pi_z^{(n)}(q^{-h_1})
\end{aligned} \tag{79}$$

with $m = 0, 1, \ldots, n$. Let

$$\begin{aligned}
\mathbf{T}^{(n+1)}(zq^{-n-1}) &= \left(\pi_z^{(n)} \otimes \pi_{\mathcal{H}}\right)\lambda^{h\otimes\mathbf{1}}\mathbf{R} \in \mathrm{End}\,\mathbb{C}^{n+1} \otimes \mathcal{H}\\
&= \lambda^{\pi^{(n)}(h)\otimes 1} L_{0M}^{(n+1)}(z/\zeta_M)\cdots L_{01}^{(n+1)}(z/\zeta_1) \quad \mathcal{H} = (\mathbb{C}^2)^{\otimes M}
\end{aligned} \tag{80}$$

with

$$L^{(n+1)}(w) = \left(\pi_w^{(n)} \otimes \pi_{z=1}^{(1)}\right)\mathbf{R} \in \mathrm{End}\,\mathbb{C}^{n+1} \otimes \mathbb{C}^2. \tag{81}$$

Note that we have labelled the fusion matrices by the dimension of their auxiliary space instead of the spin. The matrices (7), (80) and (81) again satisfy the Yang–Baxter relation. The matrix elements of (81) w.r.t. the representation $\pi_{z=1}^{(1)}$ are explicitly given by

$$\begin{aligned}
\langle 0|L^{(n+1)}(w)|0\rangle &= \rho_+\pi^{(n)}\left(q^{\frac{h}{2}}\right) - \rho_-\pi^{(n)}\left(q^{-\frac{h}{2}}\right)\\
\langle 0|L^{(n+1)}(w)|1\rangle &= \rho_+(q - q^{-1})\pi^{(n)}\left(q^{\frac{h}{2}}\right)\pi^{(n)}(f_1)\\
\langle 1|L^{(n+1)}(w)|0\rangle &= \rho_-(q - q^{-1})\pi^{(n)}(e_1)\pi^{(n)}\left(q^{-\frac{h}{2}}\right)\\
\langle 1|L^{(n+1)}(w)|1\rangle &= \rho_+\pi^{(n)}\left(q^{-\frac{h}{2}}\right) - \rho_-\pi^{(n)}\left(q^{\frac{h}{2}}\right)
\end{aligned} \tag{82}$$

with the coefficients $\rho_\pm$ satisfying the constraint $\rho_+/\rho_- = wq$. From the following non-split exact sequence describing the decomposition of evaluation representations of the quantum loop algebra $U_q(\widetilde{sl}_2)$ [21]:

$$0 \to \pi_{w'}^{(n-1)} \xhookrightarrow{\iota} \pi_w^{(n)} \otimes \pi_z^{(1)} \xrightarrow{\tau} \pi_{w''}^{(n+1)} \to 0 \qquad w = w'q^{-1} = w''q = zq^{n+1} \tag{83}$$

one derives a functional relation, known as the fusion hierarchy (see, e.g., [27, 28, 19, 30] for an alternative form of the fusion equation),

$$T^{(n)}(z)T^{(2)}(zq^{-2}) = T^{(n+1)}(zq^{-2}) \prod_{m=1}^{M} (zq^2/\zeta_m - 1) + T^{(n-1)}(zq^2) \prod_{m=1}^{M} (z/\zeta_m - 1)\,. \tag{84}$$

Here

$$T^{(n+1)}(zq^{-n-1}) = \big(\operatorname*{Tr}_{\pi_z^{(n)}} \otimes 1\big)\mathbf{T}^{(n+1)}(zq^{-n-1}) \in \operatorname{End}\mathcal{H}. \tag{85}$$

The coefficients in (82) have been chosen as

$$\rho_+ = wq \qquad \rho_- = 1 \tag{86}$$

and we have identified

$$T^{(2)}(zq^{-2}) \equiv q^{-\frac{M}{2}} T(z) \prod_{m=1}^{M} (zq^2/\zeta_m - 1) \qquad T^{(1)}(z) \equiv \prod_{m=1}^{M} (zq^2/\zeta_m - 1). \tag{87}$$

The coefficient functions in (84) have been calculated using the explicit form of the inclusion and projection map in (83),

$$0\pi^{(n-1)} \ni |m\rangle \xhookrightarrow{\iota} |m\rangle' = [n-m]|m\rangle \otimes |1\rangle - q^{n-m}[m+1]|m+1\rangle \otimes |0\rangle$$

and

$$|m\rangle'' = \frac{[n+1]}{[n-m+1]}|m\rangle \otimes |0\rangle \xrightarrow{\tau} |m\rangle \in \pi^{(n+1)}.$$

Inserting the explicit expression for the eigenvalues of the six-vertex transfer matrix from the algebraic Bethe ansatz (40) we obtain the following formula for the eigenvalues of the fusion matrices associated with the Bethe states (37):

$$T^{(n)}(z) = \lambda^{-n-1} q^{(n+1)S^z} P_B(z) P_B(zq^{2n}) \sum_{\ell=1}^{n} \lambda^{2\ell} q^{-2\ell S^z} \frac{\prod_{m=1}^{M} (zq^{2\ell}/\zeta_m - 1)}{P_B(zq^{2\ell}) P_B(zq^{2\ell-2})}. \tag{88}$$

Here we have again denoted eigenvalues and operators by the same symbol. P_B denotes the Bethe polynomial defined in (39) whose zeroes are solutions to the Bethe ansatz equations (38). The proof follows via induction and is straightforward. Note that the eigenvalues (88) are polynomials in z. The rational function in (88) only has poles at $z = z_j, z_j q^{2n}$ which are cancelled by the factor $P_B(z)P_B(zq^{2n})$ in front of the sum. The residue of the remaining poles is zero due to the Bethe ansatz equations.

Away from a root of unity we can already connect the fusion hierarchy with the auxiliary matrices (75) and (76). A straightforward calculation proves for the eigenvalues the relation

$$\lambda^{-n} q^{nS^z} Q^+(zq^{2n}) Q^-(z) - \lambda^n q^{-nS^z} Q^+(z) Q^-(zq^{2n}) = \frac{(-1)^M \lambda^{-2} q^{2S^z}}{\lambda^{-1} q^{S^z} - \lambda q^{-S^z}} T^{(n)}(z) \tag{89}$$

which corresponds to formula (4.1) in [17]. The above functional equation not only holds for the eigenvalues but also for the operators provided the Bethe states constitute a complete basis of the quantum space.

5.1. Degeneracies at roots of unity

The discussion of the fusion matrices has so far applied in the case of 'generic' q (i.e. the deformation parameter being either a root of unity or not) and finite solutions to the Bethe ansatz equations entering the eigenvalues via the Bethe polynomial (39). We now specialize in the case $q^{N'} = \pm 1$ and start by showing that the fusion matrices (85) exhibit the same infinite-dimensional non-Abelian symmetries as the transfer matrix.

Recall from [3, 5] that the symmetry generators for the XXZ spin-chain are given by ($i = 0, 1$)

$$E_i^{(N')} = \lim_{q' \to q} \bigotimes_{m=1}^{M} \pi^{(1)}_{\zeta_m} \Delta^{(M)}\big(e_i^{N'}\big) \big/ [N']_{q'}! \tag{90}$$

$$F_i^{(N')} = \lim_{q' \to q} \bigotimes_{m=1}^{M} \pi^{(1)}_{\zeta_m} \Delta^{(M)}\big(f_i^{N'}\big) \big/ [N']_{q'}!. \tag{91}$$

Here $\Delta^{(M)} = (1 \otimes \Delta)\Delta^{(M-1)}$ denotes the M-fold coproduct with $\Delta^{(2)} \equiv \Delta$ and q' is some number with $q'^{N'} \neq \pm 1$. The above form of the symmetry generators of $U(\widetilde{sl}_2)$ is restricted to the commensurate spin-sectors $2S^z = 0 \bmod N$ when $\lambda = 1$ [3] but extends for the subalgebras $U(b_{\mp})$ to all spin-sectors when $\lambda = q^{\pm S^z}$ [5]. The extension of the loop symmetry from the fusion degree $n = 2$ to arbitrary n now simply follows by induction from the recursion relation (84). Thus, the eigenspaces of the fusion matrices also organize into multiplets containing states whose total spin differs by multiples of N'.

Remark. In a similar manner one proves the loop symmetry of the higher-spin six-vertex models. The only major difference lies in the choice for the representation $\pi_{\mathcal{H}}$ defining the quantum space which changes to $\pi_{\mathcal{H}} = \bigotimes_{m=1}^{M} \pi^{(n)}_{\zeta_m}$. An explicit calculation shows that the analogue of the operators (90) is well defined. Adopting the proof in [5] for the loop symmetry of the 'fundamental' fusion matrix $T^{(2)}$ one shows again by induction that $T^{(n+1)}$ exhibits the same symmetries. This provides an alternative proof to that given in [25].

5.2. Spectrum of the fusion matrices at $q^N = 1$

In order to elucidate the spectrum of the fusion matrices at roots of unity let us discuss the limit $q' \to q$ from transcendental or irrational q' to q being a root of unity. In order to simplify the notation we will often denote this limit by the symbol $q^N \to 1$ in the following. In order to perform the root of unity limit we need the explicit dependence of the Bethe roots z_i on the arbitrary deformation parameter q'. Unfortunately, this dependence is in general not known. However, for small chains up to the size $M \leqslant 8$ one can compute some Bethe roots analytically. According to these examples the following scenarios might occur in the root of unity limit:

1. A Bethe root tends to zero,

$$\lim_{q^N \to 1} z_i = 0. \tag{92}$$

We shall denote the number of such Bethe roots by n_0.

2. A Bethe root tends to infinity,

$$\lim_{q^N \to 1} z_i = \infty. \tag{93}$$

We shall denote the number of infinite roots by n_∞.

3. There are N' Bethe roots which form a complete string

$$\lim_{q^N\to 1}\big(z_{i_0}, z_{i_1}, \ldots, z_{i_{N'-1}}\big) = \big(z_{i_0}, z_{i_1} = z_{i_0}q^2, \ldots, z_{i_{N'-1}} = z_{i_0}q^{2N'-2}\big). \tag{94}$$

   Note that these complete strings obtained in the root of unity limit do not coincide with the zeroes of the classical Drinfeld polynomial (58) contained in the spectrum of the auxiliary matrices.

Because of these three possibilities we deduce that taking the root of unity limit of the Bethe polynomial (39) is not the same as the Bethe polynomial containing the finite solutions of the Bethe ansatz equations *at* a root of unity, i.e.

$$\lim_{q'\to q} P_B(z, q') \neq P_B(z, q) \quad \text{with} \quad q^{N'} = \pm 1.$$

In the presence of vanishing Bethe roots the root of unity limit of the Bethe polynomial might not even be well defined. However, the eigenvalues of the fusion and transfer matrices only contain ratios of these polynomials. From the simple identities

$$\lim_{z_i\to 0} \frac{(z - z_i)(zq^{2n} - z_i)}{(zq^{2\ell} - z_i)(zq^{2\ell-2} - z_i)} = q^{2n-4\ell+2} \tag{95}$$

and

$$\lim_{z_i\to \infty} \frac{(z - z_i)(zq^{2n} - z_i)}{(zq^{2\ell} - z_i)(zq^{2\ell-2} - z_i)} = 1 \tag{96}$$

we infer that the spectrum of the fusion matrices at a root unity changes to

$$\lim_{q^N\to 1} T^{(n)}(z) = \lambda^{-n-1}q^{(n+1)s}P_B(z)P_B(zq^{2n})\sum_{\ell=1}^{n}\lambda^{2\ell}q^{-2\ell s}\frac{\prod_{m=1}^{M}(zq^{2\ell}/\zeta_m - 1)}{P_B(zq^{2\ell})P_B(zq^{2\ell-2})}. \tag{97}$$

Here we have set

$$s = 2n_0 + S^z \bmod N'. \tag{98}$$

Again we remind the reader that P_B is now the 'reduced' Bethe polynomial, i.e. it only contains the *finite* Bethe roots at $q^N = 1$ and no complete strings. From the identity (97) it follows by a direct calculation that one has the following simplification of the fusion hierarchy in terms of the eigenvalues:

$$\lim_{q^N\to 1} T^{(N'+1)}(z) = (\lambda^{-N'}q^{N's} + \lambda^{N'}q^{-N's})\prod_{m=1}^{M}(zq^2/\zeta_m - 1) + \lim_{q^N\to 1} T^{(N'-1)}(zq^2). \tag{99}$$

This formula has been reported for $\lambda = 1$, $\zeta_m = 1$ before in the literature [31, 29, 14].

5.3. Connection with auxiliary matrices at roots of 1

We will now argue that the fusion matrix of degree N' at $q^{N'} = \pm 1$ can be identified with a special limit of the auxiliary matrix (49). First, we observe that the corresponding auxiliary spaces have the same dimension. From the representation theory of quantum groups at roots of unity [32, 33] one now deduces the following. Restrict the respective evaluation representations (18) and (79) of the quantum loop algebra to the finite subalgebra $U_q(sl_2)$. If the values of the central elements in the respective $U_q(sl_2)$-representations determining the auxiliary spaces are equal then they must be isomorphic. The values of the central elements at a root of unity are

$$T^{(N')}: \quad \pi^{(N')}\big(e_1^{N'}\big) = \pi^{(N')}\big(f_1^{N'}\big) = 0 \qquad \pi^{(N')}(q^h)^{N'} = q^{N'(N'-1)}$$
$$Q_\mu: \quad \pi^{\mu}\big(e_1^{N'}\big) = \pi^{\mu}\big(f_1^{N'}\big) = 0 \qquad \pi^{\mu}(q^h)^{N'} = q^{-N'}\mu^{-N'}$$

and for the Casimir $c = q^{h+1} + q^{-h-1} + (q - q^{-1})^2 f e$ one finds

$$T^{(N')}: \quad \pi^{(N')}(c) = q^{N'} + q^{-N'}$$
$$Q_\mu: \quad \pi^{\mu}(c) = \mu + \mu^{-1}.$$

Hence, in the limit $\mu \to q^{N'} = \pm 1$ both representations become isomorphic and one can therefore construct a gauge transformation in the auxiliary space rendering the fusion and auxiliary matrix equal (see [1] for details). Thus, we conclude

$$\lim_{q^N \to 1} T^{(N')}(z) = \lim_{\mu \to q^{N'}} Q_\mu(z/\mu). \tag{100}$$

Note that in this limit the auxiliary matrix therefore also becomes degenerate. Nevertheless, we can check the eigenvalues for consistency by comparing the expressions for the eigenvalues derived from the algebraic Bethe ansatz with those from the fusion hierarchy. One finds for the eigenvalues of the fusion matrix

$$\lim_{q^N \to 1} T^{(N')}(z) = \lambda^{-N'-1} q^{(N'+1)s} P_B(z) P_B(zq^{2N'}) \sum_{\ell=1}^{N'} \frac{\lambda^{2\ell} q^{-2\ell s} \prod_{m=1}^{M} (zq^{2\ell}/\zeta_m - 1)}{P_B(zq^{2\ell}) P_B(zq^{2\ell-2})}. \tag{101}$$

In the commensurate sectors $2S^z = 0 \bmod N$ where $n_0 = n_\infty = 0$ we can set $s = 0$ in (101). Then the spectrum of the auxiliary matrices Q_μ derived for finite Bethe states at a root of unity coincides for $\mu = q^{N'} = \pm 1$ with that of the fusion matrix. Outside the commensurate sectors it has been argued (and proved for $N = 3$) in [2] that the auxiliary matrices (49) have eigenvalues of the form (57). When comparing with formula (11) in [2] one has to make the replacement $\bar{n}_\infty \to n_\infty$ in the notation. Here P_S denotes the previously discussed polynomial (58) whose zeroes are fixed by the representation theory of the loop algebra up to a factor $\mu^{2N'}$,

$$q^{N'} = 1: \quad \mathcal{N}_\mu z^{\bar{n}_\infty} P_S(z^{N'}, \mu = q^{N'}) = q^{(N'+1)s} \sum_{\ell \in \mathbb{Z}_{N'}} \frac{q^{-2\ell s} \prod_{m=1}^{M} (zq^{2\ell}/\zeta_m - 1)}{P_B(zq^{2\ell}) P_B(zq^{2\ell-2})}. \tag{102}$$

The power of the monomial must be an integer by construction of the auxiliary matrix and the normalization chosen in (19). Interestingly, numerical examples show that vanishing and infinite Bethe roots seem to be absent for all spin-sectors when $\lambda = q^{\pm S^z}$, i.e. for the boundary conditions where the symmetry generators of the loop algebra are known for all spin-sectors [5]. There appears to be a close connection between the form of the symmetry algebra generators and the presence of infinite or vanishing Bethe roots. This might hold the clue for constructing the symmetry algebra outside the commensurate sectors when $\lambda = 1$. Further investigation is needed to clarify this point.

6. Conclusions

In this paper we have connected two known alternative methods of solving integrable models, the algebraic Bethe ansatz and Baxter's concept of auxiliary matrices. Within the community of integrable models the notion of auxiliary matrices has historically received less attention than the Bethe ansatz, even though it applies to a wider range of models. Unlike the Bethe ansatz, auxiliary matrices do not rely on the existence of a pseudo-vacuum or spin conservation; see the discussion in section 3. One possible reason that auxiliary matrices have not earlier been investigated in more detail, might be that Baxter's construction procedure [15] leads to a final auxiliary matrix whose algebraic structure is not of the simple form (49). Note also that Baxter's auxiliary matrix for $\lambda = 1$ is limited to the case of even spin-chains, $M \in 2\mathbb{N}$, when q is not a root of unity and to $M \bmod N \in 2\mathbb{N}$ when $q^N = 1$; see condition (9.8.16) in [15].

For a construction of Baxter's auxiliary matrices at more general boundary conditions see, e.g., [34].

In contrast the construction of auxiliary matrices using intertwiners of quantum groups yields simpler algebraic expressions, has no restrictions on the length of the spin-chain and, most importantly, allows for maintaining the Yang–Baxter relation (15). It is the last fact which puts us in the position to link the formalism of the algebraic Bethe ansatz with the Q-operator or auxiliary matrix. Furthermore, the use of representation theory in the construction of Q-operators enables one to make direct contact with the underlying algebraic structure of the integrable model, the quantum loop algebra. The auxiliary matrices discussed in this paper are therefore a key link in understanding the relation between the Bethe ansatz and the representation theory of quantum groups.

The importance of this aspect of the Q-matrix is particularly highlighted in the root of unity case. As pointed out in [2] the auxiliary matrices constructed at a root of unity [1] might serve as an efficient tool to analyse the irreducible representations of the loop algebra symmetry. In our investigation of the spectrum of the auxiliary matrices via the Bethe ansatz as well as the fusion hierarchy we came across the polynomial (cf formula (58) in the text)

$$\mathcal{N}_{\mu=1} P_S(z^{N'}, \mu = 1) = \sum_{k\in\mathbb{Z}_{N'}} \frac{\lambda^{-2k} q^{2kS^z} (zq^{-2k} - 1)^M}{P_B(zq^{-2k}) P_B(zq^{-2k-2})} \qquad q^{N'} = \pm 1. \tag{103}$$

Recall that the Bethe ansatz equations are sufficient to ensure that the right-hand side of this identity is a polynomial. This expression has been conjectured to be the classical Drinfeld polynomial [23]. The latter describes the irreducible representation of the loop algebra spanning the degenerate eigenspace of the transfer matrix; see [2] for concrete examples.

What is the benefit of the identification? The symmetry algebra can be used to compute the Drinfeld polynomial, i.e. the left-hand side of equation (103). We can then use the above identity to extract the Bethe roots for each degenerate eigenspace of the transfer matrix. At the moment this identification still awaits proof as well as its extension to the quasi-periodic boundary conditions $\lambda = q^{\pm S^z}$ investigated in [5]. The results of this paper provide a further step towards this direction.

Away from a root of unity our interest has been to make contact with the findings in [16, 17] for the Coulomb gas formalism of conformal field theory. This also provided the motivation to perform the algebraic Bethe ansatz computation of the spectrum of the Q-operators for an arbitrary quantum space in order to accommodate the CFT setup. For the XXZ spin-chain we started from the results obtained in [19] to analyse the spectrum of the auxiliary matrix and to find the analogue of the functional equations reported in [17] for the six-vertex model. The noteworthy difference between the constructions in [17] and [19] is the occurrence of free parameters in the representation spanning the auxiliary space of the Q-operator; see definition (23). We saw that the introduction of such parameters is natural in light of the representation theoretic background (26) of the TQ-equation (28); see also [19].

What is the role of these parameters with hindsight of the spectrum of the six-vertex model? Our result for the eigenvalues (66) and (72) showed that the parameter r_0 in (23) is needed to break spin-reversal symmetry, while the parameters $r_{1,2}$ simply reflect the polynomial structure of the eigenvalues of the auxiliary matrix in the spectral variable z. One of them is needed to truncate the auxiliary space in order to achieve convergence when q is not a root of unity. The freedom in choosing the remaining parameter can be used to decompose the eigenvalue (66) into the polynomials (75) and (76); see also (77). This decomposition has been observed in [17] in the context of CFT (see equation (4.10) therein) and was the starting point for the formulation of a series of functional equations all of which we recovered in the case of the XXZ spin-chain; see, for example, equation (89) in the text. As these functional relations have been the starting point for connecting the spectrum of the auxiliary matrices with ordinary differential equations [35], it is natural to ask whether this can also be achieved for the XXZ spin-chain. So far partial results exist at some roots of unity only [36].

Convergence problems did not arise in the root of unity case and one might ask about the root of unity limit of the auxiliary matrix (66). While we did not pursue this issue in detail it seems at first sight plausible that in this limit the infinite-dimensional representation splits up into an infinite number of finite-dimensional subrepresentations. Reducing the trace of the monodromy matrix (13) to such a subrepresentation one might be lead to the conclusion that one ends up with the auxiliary matrices constructed *at* a root of unity in [1]. This is not true. The full set of auxiliary matrices constructed in [1] does not preserve the total spin S^z. Here we only dealt with a subset of the auxiliary matrices available at a root of unity. These additional Q-operators also contain free parameters but their nature is different from those in (23). They reflect the enhanced symmetry of the XXZ spin-chain at rational coupling, i.e. $q^{N'} = \pm 1$. The full scope of this symmetry is not accessible via the Bethe ansatz whence further work is required to determine the spectrum of all auxiliary matrices in [1] as well as the spectrum inside the degenerate eigenspaces; see our discussion in section 4.1.2.

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Appendix. The spectrum of Q. Proof for $n_B = 2, 3$

We outline the main steps in proving the conjecture (46) for the Bethe states (37) with $n_B = 2, 3$. We will make use of the following well-known commutation relation of the Yang–Baxter algebra:

$$A_1 B_2 = \frac{1}{b_{21}} B_2 A_1 - \frac{c_{21}}{b_{21}} B_1 A_2 \qquad D_1 B_2 = \frac{1}{b_{12}} B_2 D_1 - \frac{c'_{12}}{b_{12}} B_1 D_2$$

$$[D_1, A_2] = \frac{c_{12}}{b_{12}} B_2 C_1 - \frac{c'_{12}}{b_{12}} B_1 C_2 \qquad [C_1, B_2] = \frac{c'_{12}}{b_{12}} (A_2 D_1 - A_1 D_2).$$

Here the indices are a shorthand notation for the dependence of the operators (9) and the Boltzmann weights (12) on the respective Bethe roots. For instance,

$$b_{ij} \equiv b(z_i/z_j), \ldots \quad \text{and} \quad A_i \equiv A(z_i), \quad B_i \equiv B(z_i), \ldots \quad \text{etc.}$$

To unburden the formulae we also introduce the symbols

$$\Lambda^i_{kl} \equiv \frac{\delta_l(w/z_i)}{\alpha_k(w/z_i)} - \frac{\beta_{l+1}(w/z_i)\gamma_l(w/z_i)}{\alpha_{l+1}(w/z_i)\alpha_k(w/z_i)} \qquad \text{and} \qquad r^i_k \equiv \frac{\beta_k(w/z_i)}{\alpha_k(w/z_i)}.$$

The case $n_B = 2$. From the relations (30), (31), (32) and (33) one obtains

$$\begin{aligned} Q_{kk} B_1 B_2 = {} & \Lambda^1_{kk} \Lambda^2_{kk} B_1 B_2 Q_{kk} \\ & + \frac{r^1_{k+1}}{b_{21}} Q_{kk+1} B_2 A_1 + \Lambda^1_{kk} r^2_{k+1} B_1 Q_{kk+1} A_2 - r^1_{k+1} \frac{c_{21}}{b_{21}} Q_{kk+1} B_1 A_2 \\ & - \frac{r^1_k}{b_{12}} Q_{k-1k} B_2 D_1 - \Lambda^1_{kk} r^2_k B_1 Q_{k-1k} D_2 + r^1_k \frac{c'_{12}}{b_{12}} Q_{k-1k} B_1 D_2 \end{aligned}$$

$$+r^1_{k+1}r^1_k Q_{k-1k+1}B_2C_1 + \Lambda^1_{kk}r^2_{k+1}r^2_k B_1 Q_{k-1k+1}C_2$$
$$+r^1_{k+1}r^1_k \frac{c'_{12}}{b_{12}} Q_{k-1k+1}(A_2D_1 - A_1D_2).$$

Moving all the B-operators past the Q-operators except for the first term, this can be rewritten as

$$\begin{aligned}
Q_{kk}B_1B_2 = {} & \Lambda^1_{kk}\Lambda^2_{kk}B_1B_2Q_{kk} + \frac{r^1_{k+1}}{b_{21}}Q_{kk+1}B_2A_1 + \frac{r^2_{k+1}}{b_{12}}Q_{kk+1}B_1A_2 \\
& - \frac{r^1_k}{b_{12}}Q_{k-1k}B_2D_1 - \frac{r^2_k}{b_{21}}Q_{k-1k}B_1D_2 - r^1_{k+2}r^2_{k+1}\frac{\Lambda^1_{kk}}{\Lambda^1_{kk+1}}Q_{kk+2}A_1A_2 \\
& - r^2_k r^1_{k-1}\frac{\Lambda^1_{kk}}{\Lambda^1_{k-1k}}Q_{k-2k}D_1D_2 + \frac{r^1_k r^2_{k+1}}{b_{12}}Q_{k-1k+1}A_2D_1 + \frac{r^1_{k+1}r^2_k}{b_{21}}Q_{k-1k+1}A_1D_2 \\
& + r^1_k r^1_{k+1}\frac{\Lambda^2_{kk}}{\Lambda^2_{kk+1}b_{12}}Q_{k-1k+1}B_2C_1 + r^2_k r^2_{k+1}\frac{\Lambda^1_{kk}}{\Lambda^1_{kk+1}b_{21}}Q_{k-1k+1}B_1C_2 \\
& - r^1_{k+2}r^2_{k+1}r^1_k\frac{\Lambda^1_{kk}}{\Lambda^1_{kk+1}}Q_{k-1k+2}C_1A_2 - r^1_{k+2}r^2_{k+1}r^2_k\frac{\Lambda^1_{kk}}{\Lambda^1_{k-1k+1}}Q_{k-1k+2}A_1C_2 \\
& + r^1_{k+1}r^2_k r^1_{k-1}\frac{\Lambda^1_{kk}}{\Lambda^1_{k-1k}}Q_{k-2k+1}C_1D_2 + r^2_{k+1}r^2_k r^1_{k-1}\frac{\Lambda^1_{kk}}{\Lambda^1_{k-1k+1}}Q_{k-2k+1}D_1C_2 \\
& - r^1_{k+2}r^2_{k+1}r^2_k r^1_{k-1}\frac{\Lambda^1_{kk}}{\Lambda^1_{k-1k+1}}Q_{k-2k+2}C_1C_2.
\end{aligned}$$

Acting with this expression on the pseudo-vacuum all the terms containing C-operators vanish. Taking the trace on both sides of the equation we then obtain

$$\begin{aligned}
\sum_k Q_{kk}B_1B_2|0\rangle_{\mathcal{H}} = {} & \sum_k \Lambda^1_{kk}\Lambda^2_{kk}B_1B_2Q_{kk}|0\rangle_{\mathcal{H}} + \sum_k r^1_{k+1}Q_{kk+1}B_2(A_1/b_{21} - D_1/b_{12})|0\rangle_{\mathcal{H}} \\
& + \sum_k \frac{r^2_{k+1}}{b_{12}}Q_{kk+1}B_1(A_2/b_{12} - D_2/b_{21})|0\rangle_{\mathcal{H}} \\
& + \sum_k \left(\frac{r^1_{k+1}r^2_k}{b_{21}}Q_{k-1k+1}A_1D_2 + \frac{r^1_k r^2_{k+1}}{b_{12}}Q_{k-1k+1}A_2D_1\right)|0\rangle_{\mathcal{H}} \\
& - \sum_k \left(r^1_{k+2}r^2_{k+1}\frac{\Lambda^1_{kk}}{\Lambda^1_{kk+1}}Q_{kk+2}A_1A_2 + r^2_k r^1_{k-1}\frac{\Lambda^1_{kk}}{\Lambda^1_{k-1k}}Q_{k-2k}D_2D_1\right)|0\rangle_{\mathcal{H}}.
\end{aligned}$$

Employing the Bethe ansatz equations (38),

$$\langle 0|A_i|0\rangle_{\mathcal{H}}b_{ij} = b_{ji}\langle 0|D_i|0\rangle_{\mathcal{H}} \qquad i,j = 1,2$$

and using the identity

$$\frac{r^1_{k+1}r^2_k}{b_{12}} + \frac{r^2_{k+1}r^1_k}{b_{21}} - r^1_{k+1}r^2_k\frac{\Lambda^1_{k-1k-1}}{\Lambda^1_{k-1k}} - r^2_{k+1}r^1_k\frac{\Lambda^1_{k+1k+1}}{\Lambda^1_{kk+1}} = 0$$

we see that the remaining terms cancel. This proves (46) for $n_B = 2$.

The case $n_B = 3$. The calculation follows the same strategy as before. The various steps can be simplified by observing that the expressions must be symmetric under any permutation of the Bethe roots. We omit the details of the computation and only provide some intermediate steps. Starting from the result for the case $n_B = 2$ one has

$$
\begin{aligned}
Q_{kk}B_1B_2B_3|0\rangle_{\mathcal{H}} = {} & \Lambda^1_{kk}\Lambda^2_{kk}B_1B_2Q_{kk}B_3|0\rangle_{\mathcal{H}} + Q_{kk+1}\left(\frac{r^2_{k+1}}{b_{32}b_{12}}B_1B_3A_2 - \frac{r^2_{k+1}c_{32}}{b_{12}b_{32}}B_1B_2A_3\right.\\
&\left. + \frac{r^1_{k+1}}{b_{21}b_{31}}B_2B_3A_1 - \frac{r^1_{k+1}c_{31}}{b_{21}b_{31}}B_2B_1A_3\right)|0\rangle_{\mathcal{H}}\\
& - Q_{k-1k}\left(\frac{r^2_k}{b_{21}b_{23}}B_1B_3D_2 - \frac{r^2_k c'_{23}}{b_{21}b_{23}}B_1B_2D_3\right.\\
&\left. + \frac{r^1_k}{b_{12}b_{13}}B_2B_3D_1 - \frac{r^1_k c'_{13}}{b_{12}b_{13}}B_2B_1D_3\right)|0\rangle_{\mathcal{H}}\\
& + Q_{k-1k+1}\frac{r^1_{k+1}r^2_k}{b_{21}}\left(\frac{1}{b_{31}b_{23}}B_3A_1D_2 - \frac{c_{31}}{b_{31}b_{23}}B_1A_3D_2\right.\\
&\left. - \frac{c'_{23}}{b_{21}b_{23}}B_2A_1D_3 + \frac{c_{21}c'_{23}}{b_{21}b_{23}}B_1A_2D_3\right)|0\rangle_{\mathcal{H}}\\
& + Q_{k-1k+1}\frac{r^2_{k+1}r^1_k}{b_{12}}\left(\frac{1}{b_{32}b_{13}}B_3A_2D_1 - \frac{c_{32}}{b_{32}b_{13}}B_2A_3D_1\right.\\
&\left. - \frac{c'_{13}}{b_{12}b_{13}}B_1A_2D_3 + \frac{c_{12}c'_{13}}{b_{12}b_{13}}B_2A_1D_3\right)|0\rangle_{\mathcal{H}}\\
& + Q_{k-1k+1}\left(\frac{\Lambda^1_{kk}c'_{23}}{\Lambda^1_{kk+1}b_{23}}r^2_{k+1}r^2_kB_1(A_3D_2 - A_2D_3)\right.\\
&\left. + \frac{\Lambda^2_{kk}c'_{13}}{\Lambda^2_{kk+1}b_{13}}r^1_{k+1}r^1_kB_2(A_3D_1 - A_1D_3)\right)|0\rangle_{\mathcal{H}}\\
& - Q_{kk+2}\frac{\Lambda^1_{kk}r^1_{k+2}r^2_{k+1}}{\Lambda^1_{kk+1}}\left(\frac{1}{b_{31}b_{32}}B_3A_1A_2 - \frac{c_{31}}{b_{31}b_{32}}B_1A_3A_2\right.\\
&\left. + \frac{c_{21}c_{32}}{b_{21}b_{32}}B_1A_2A_3 - \frac{c_{32}}{b_{32}b_{21}}B_2A_1A_3\right)|0\rangle_{\mathcal{H}}\\
& - Q_{k-2k}\frac{\Lambda^1_{kk}r^1_{k-1}r^2_k}{\Lambda^1_{k-1k}}\left(\frac{1}{b_{13}b_{23}}B_3D_1D_2 - \frac{c'_{13}}{b_{13}b_{23}}B_1D_3D_2\right.\\
&\left. + \frac{c'_{12}c'_{23}}{b_{12}b_{23}}B_1D_2D_3 - \frac{c'_{23}}{b_{23}b_{12}}B_2D_1D_3\right)|0\rangle_{\mathcal{H}}\\
& - Q_{k-1k+2}\left(\frac{\Lambda^2_{kk}r^1_kr^2_kr^2_{k+2}c'_{23}}{\Lambda^2_{kk+1}b_{21}b_{23}}A_1(A_3D_2 - A_2D_3)\right.\\
&\left. + \frac{\Lambda^1_{kk}r^2_kr^1_kr^1_{k+2}c'_{13}}{\Lambda^1_{kk+1}b_{12}b_{13}}A_2(A_3D_1 - A_1D_3)\right)|0\rangle_{\mathcal{H}}\\
& + Q_{k-2k+1}\left(\frac{\Lambda^2_{kk}r^2_{k-1}r^1_kr^2_{k+1}c'_{23}}{\Lambda^2_{k-1k}b_{12}b_{23}}D_1(A_3D_2 - A_2D_3)\right.\\
&\left. + \frac{\Lambda^1_{kk}r^1_{k-1}r^2_kr^1_{k+1}c'_{13}}{\Lambda^1_{k-1k}b_{21}b_{13}}D_2(A_3D_1 - A_1D_3)\right)|0\rangle_{\mathcal{H}}.
\end{aligned}
$$

Here we have dropped all terms where C acts on the pseudo-vacuum first and those which contain more C than B-operators. The last identity is further simplified to

$$
\begin{aligned}
Q_{kk}B_1B_2B_3|0\rangle_{\mathcal{H}} &= \Lambda^1_{kk}\Lambda^2_{kk}\Lambda^3_{kk}B_1B_2B_3Q_{kk}|0\rangle_{\mathcal{H}} \\
&+ Q_{kk+1}\left(\frac{r^1_{k+1}}{b_{21}b_{31}}B_2B_3A_1 + \frac{r^2_{k+1}}{b_{12}b_{32}}B_1B_3A_2 + \frac{r^3_{k+1}}{b_{13}b_{23}}B_1B_2A_3\right)|0\rangle_{\mathcal{H}} \\
&- Q_{k-1k}\left(\frac{r^1_{k}}{b_{13}b_{12}}B_2B_3D_1 + \frac{r^2_{k}}{b_{21}b_{23}}B_1B_3D_2 + \frac{r^3_{k}}{b_{31}b_{32}}B_1B_2D_3\right)|0\rangle_{\mathcal{H}} \\
&+ Q_{k-1k+1}\left(\frac{r^1_{k+1}r^2_k}{b_{21}b_{31}b_{23}}B_3A_1D_2 + \frac{r^1_k r^2_{k+1}}{b_{12}b_{32}b_{13}}B_3A_2D_1 + \cdots\right)|0\rangle_{\mathcal{H}} \\
&- Q_{kk+2}\left(\frac{\Lambda^1_{kk}r^1_{k+2}r^2_{k+1}}{\Lambda^1_{kk+1}b_{31}b_{32}}B_3A_1A_2 + \cdots\right)|0\rangle_{\mathcal{H}} \\
&- Q_{k-2k}\left(\frac{\Lambda^1_{kk}r^1_{k-1}r^2_{k}}{\Lambda^1_{k-1k}b_{13}b_{23}}B_3D_1D_2 + \cdots\right)|0\rangle_{\mathcal{H}} \\
&- Q_{k-1k+2}\left(\frac{\Lambda^1_{kk}r^3_k r^2_{k+1}r^1_{k+2}}{\Lambda^1_{kk+1}b_{31}b_{32}}A_1A_2D_3 + \cdots\right)|0\rangle_{\mathcal{H}} \\
&+ Q_{k-2k+1}\left(\frac{\Lambda^1_{kk}r^1_{k-1}r^2_k r^3_{k+1}}{\Lambda^1_{k-1k}b_{13}b_{23}}D_1D_2A_3 + \cdots\right)|0\rangle_{\mathcal{H}} \\
&+ \frac{\Lambda^1_{kk}\Lambda^2_{kk}}{\Lambda^1_{kk+1}\Lambda^2_{kk+2}}r^3_{k+1}r^1_{k+2}r^2_{k+3}Q_{kk+3}A_1A_2A_3|0\rangle_{\mathcal{H}} \\
&- \frac{\Lambda^1_{kk}\Lambda^2_{kk}}{\Lambda^1_{k-1k}\Lambda^2_{k-2k}}r^3_k r^1_{k-1}r^2_{k-2}Q_{k-3k}D_1D_2D_3|0\rangle_{\mathcal{H}}.
\end{aligned}
$$

The omitted terms in the parentheses are obtained by symmetrization w.r.t. the Bethe roots. Again one computes that upon taking the trace on both sides of the equation and invoking the Bethe ansatz equations,

$$
\langle 0|A_i|0\rangle_{\mathcal{H}}\prod_{j\neq i} b_{ij} = \langle 0|D_i|0\rangle_{\mathcal{H}}\prod_{j\neq i} b_{ji}
$$

all terms vanish except the first one. This proves (46) for $n_B = 3$. We leave the case for general n_B to a future calculation. Here we shall be content with supporting the conjecture for $n_B > 3$ by making contact with various functional equations for the eigenvalues; see (55), (61), (89), (100) and (101).

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